Lecture 2 The Solow Growth Model

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Solow Growth Model

近因

- Develop a simple framework for the proximate causes and the
机制 mechanics of economic growth and cross-country income differences.

NX= 0

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, \cdots$
- Economy is inhabited by a large number of households, and for now households will not be optimizing. 没有明确最大化问题
- This is the main difference between the Solow model and the neoclassical growth model.
- Assume all households are identical, so the economy admits a representative household. 同一的

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income

边际消费倾向

- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a representative firm, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y_t = F(K_t, L_t, A_t)$$

最终产品可直接用于再生产

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- A_t is a shifter of the production function. Broad notion of technology.

广义概念

- Major assumption: technology is free; it is publicly available as a non-excludable, non-rival good. 无竞争、无排他（公共物品）

Some Assumptions

- Assumption 1 (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale) The production function F : $R^3_+ \to R_+$ is twice continuously differentiable in K and L, and satisfies:

边际产出递减
规模效应不变

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K} < 0, F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L} < 0.$$

  Moreover, F exhibits constant returns to scale in K and L.

- Assume F exhibits constant returns to scale in K and L. I.e., it is linearly homogeneous (homogeneous of degree 1) in these two variables.

一次齐次

Review

- **定义[n次齐次]：** 函数f是在x和y上n次齐次的，当且仅当对于任意的正实数λ

$$f(\lambda x, \lambda y, z) = \lambda^n f(x, y, z).$$

- **定理[欧拉定理]：** 如果函数f是在x和y上是连续可微的，用f_x, f_y分别表示函数在x, y上的偏导数，并且函数f是在x和y上是n次齐次的话，那么对于任意的实数x,y:

$$nf(x, y, z) = f_x(x, y, z)x + f_y(x, y, z)y.$$

并且，f_x, f_y本身是在x和y上n-1次齐次的。

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive. 竞争市场
- Households own all of the labor, which they supply inelastically. 劳动力供应无弹性
- Endowment of labor in the economy, $\bar{L}_t$, and all of this will be supplied regardless of the price.
- The labor market clearing condition can then be expressed as:

$$ {}^{D}L_t = \bar{L}_t{}^{S} $$

for all t, where L_t denotes the demand for labor (and also the level of employment).

More generally, should be written in complementary slackness form. In particular, let the wage rate at time t be w_t, then the labor market clearing condition takes the form

$$L_t \leq \bar{L}_t, w_t \geq 0, \quad and \quad (L_t - \bar{L}_t)w_t = 0.$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive. $w_t > 0$
- Households also own the capital stock of the economy and rent it to firms. Take initial holdings, K_0, as given. 起始资本存量
- Denote the rental price of capital at time t be R_t.
- Capital market clearing condition:

$$K_t^s = K_t^d$$

- Assume capital depreciates at the rate δ. 资本折旧
- Then, the interest rate faced by the household will be $r_t = R_t - \delta$.

Firm Optimization I

- Only need to consider the problem of a representative firm:

利润最大化

$$\max_{K_t \geq 0, L_t \geq 0} F(K_t, L_t, A_t) - R_t K_t - w_t L_t$$

无不可逆投资或调整成本

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem. 只考虑当期的静态问题
- Equivalently, cost minimization problem.
- Features worth noting:
  - Problem is set up in terms of aggregate variables. 加合过
  - Nothing multiplying the F term, price of the final good has normalized to 1. 最终商品价格标准化
  - Already imposes competitive factor markets: firm is taking as given w_t and R_t. 竞争要素市场

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w_t = F_L(K_t, L_t, A_t)$$
$$R_t = F_K(K_t, L_t, A_t)$$

- Note also that in equations above, we used K_t and L_t, the amount of capital and labor used by firms.
- In fact, solving for K_t and L_t, we can derive the capital and labor demands of firms in this economy at rental prices R_t and w_t.
- Thus we could have used K_t^d instead of K_t, but this additional notation is not necessary.

Firm Optimization III

$$Y_t = F(L_t, K_t) = F_L \cdot L_t + F_K \cdot K_t = w_t L_t + R_t \cdot K_t.$$

- **Proposition**
  Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y_t = w_t L_t + R_t K_t$$

- Proof: Follows immediately from Euler Theorem for the case of m = 1, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

$$\text{Labor Share} = \frac{w_t L_t}{Y_t}$$

$$\text{Capital Share} = \frac{R_t K_t}{Y_t}$$

Second Key Assumption

- Assumption 2 (Inada conditions) F satisfies the Inada conditions

$$\lim_{K\to 0} F_K(\cdot) = \infty, \lim_{K\to\infty} F_K(\cdot) = 0,$$

for all $L > 0$ all A

$$\lim_{L\to 0} F_L(\cdot) = \infty, \lim_{L\to\infty} F_L(\cdot) = 0.$$

for all $K > 0$ all A.

- Important in ensuring the existence of interior equilibria.

Production Functions

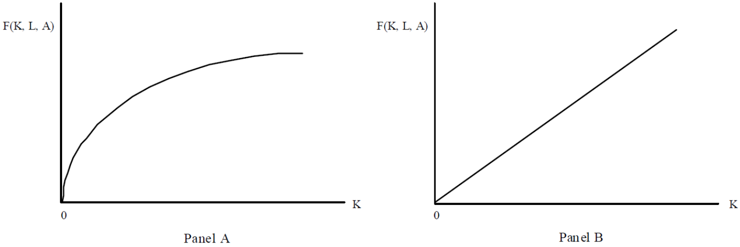


Figure: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ, so

$$K_{t+1} = (1 - \delta)K_t + I_t.$$

  where I_t is the investment at time t.
- From national income accounting for a closed economy,

$$Y_t = C_t + I_t.$$

- Behavioral rule of the constant saving rate simplifies the structure of equilibrium considerably.

Fundamental Law of Motion of the Solow Model II

- Since the economy is closed (and there is no government spending),
$$S_t = I_t = Y_t - C_t.$$
- Individuals are assumed to save a constant fraction s of their income,
$$S_t = sY_t,$$
$$C_t = (1 - s)Y_t.$$
- Implies that the supply of capital resulting from households' behavior can be expressed as
$$K_{t+1}^s = (1 - \delta)K_t + S_t = (1 - \delta)K_t + sY_t.$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies

$$K_t^s = K_t$$

- We also have

$$L_t = \bar{L}_t.$$

- Combining these market clearing conditions with the law of motion for capital in previous slides and the production function.

$$K_{t+1}^s = (1-\delta)K_t + sF(K_t, L_t, A_t).$$

- Equilibrium of the Solow growth model is described by this equation together with laws of motion for L_t and A_t.

Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.
- In the basic Solow model for a given sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K_t, Y_t, C_t, w_t, R_t\}_{t=0}^{\infty}$ such that K_t satisfies

  Household

  $$K_{t+1} \leq (1-\delta)K_t + F(K_t, L_t, A_t) - C_t;$$

  Y_t is given by $Y_t = F(K_t, L_t, A_t)$; C_t is given by $C_t = (1-s)Y_t$; w_t is given by $w_t = F_L(K_t, L_t, A_t)$; R_t is given by $R_t = F_K(K_t, L_t, A_t)$.
- Note an equilibrium is defined as an entire path of allocations and prices: not a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
  - There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L_t = L$. 无人口增长和技术进步
  - No technological progress, so that $A_t = A$.
- Define the capital-labor ratio of the economy as

$$k_t \equiv \frac{K_t}{L}.$$

- Using the constant returns to scale assumption, we can express output (income) per capita,

$$y_t \equiv \frac{Y_t}{L}.$$

as

$$y_t \equiv \frac{Y_t}{L} = F(\frac{K_t}{L}, 1, A) \equiv f(k_t).$$

Equilibrium Without Population Growth and Technological Progress II

- Note that $f(k)$ here depends on A, so I could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$R_t = f'(k_t)$$

人均形式

and

$$\begin{aligned} w_t &= \frac{Y_t - R_t K_t}{L_t} \\ &= y_t - R_t k_t \\ &= f(k_t) - f'(k_t) k_t \end{aligned}$$

$w_t = f(k_t) - f'(k_t)k_t$

$R_t = f'(k_t)$

(remember that

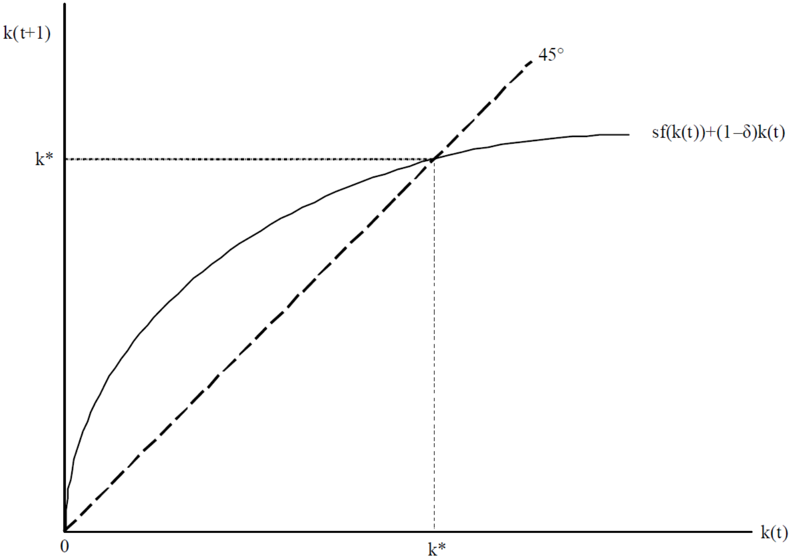
$$Y_t = F(K_t, L_t, A_t) = w_t L_t + R_t K_t.)$$

- Both are positive from Assumption 1.

Equilibrium Without Population Growth and Technological Progress III

- The per capita representation of the aggregate production function:
$$k_{t+1} = (1-\delta)k_t + sf(k_t).$$
- It can be referred to as the equilibrium difference equation of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio k_t.
- **Definition** A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k_t = k^*$ for all t.
- The economy will tend to this steady state equilibrium over time.

Steady-State Capital-Labor Ratio



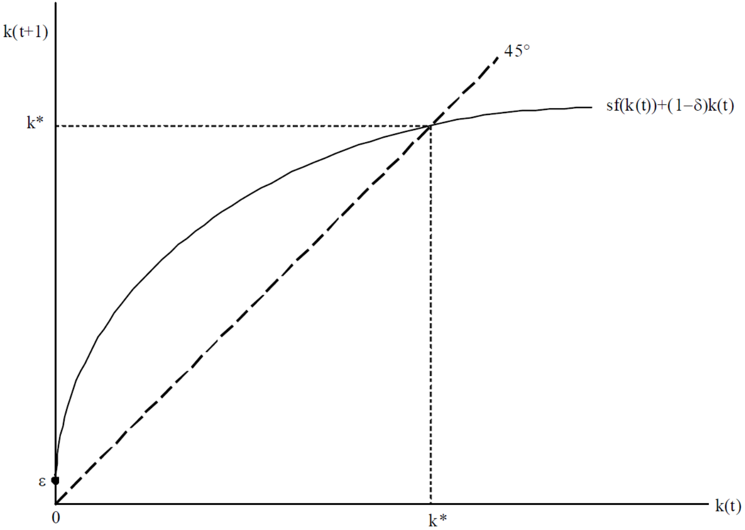
Equilibrium Without Population Growth and Technological Progress III

- Thick curve represents the law of motion for capital per capita, the dashed line corresponds to the 45 degree line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*,

$$k^* = (1-\delta)k^* + sf(k^*)$$
$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}$$

- There is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$. 另一交点为0
- Will ignore this intersection throughout:
  - If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium.
  - This intersection, even when it exists, is an unstable point.
  - It has no economic interest for us.

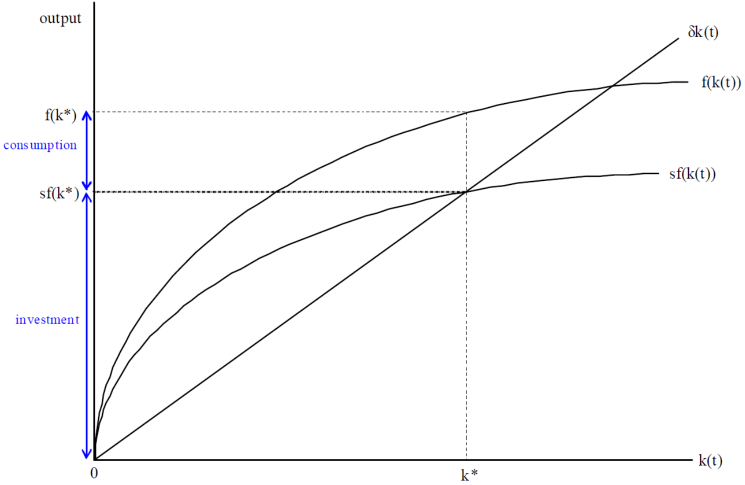
Equilibrium Without Population Growth and Technological Progress III



Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. Useful because:
  - Depicts the levels of consumption and investment in a single figure.
  - Emphasizes the steady-state equilibrium sets investment, $sf(k)$, equal to the amount of capital that needs to be "replenished", δk.

Consumption and Investment in Steady State



Equilibrium Without Population Growth and Technological Progress V

- Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s},$$

per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*)$$

Proof

- The preceding argument establishes that any k^* satisfies $\frac{f(k^*)}{k^*} = \frac{\delta}{s}$ is a steady state. $\lim_{k\to 0} \frac{f(k)}{k} \overset{0}{\underset{0}{=}} \lim_{k\to 0} f'(k) = \infty$
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule), $\lim_{k\to 0} f(k)/k = \infty$ and $\lim_{k\to\infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 11, so by the Intermediate Value Theorem there exists k^* such that $\frac{f(k^*)}{k^*} = \frac{\delta}{s}$ is satisfied. 介值定理
- To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives 单调性

$$\frac{\partial f(k)/k}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0,$$

where the last equality uses

$$w_t = f(k_t) - f'(k_t)k_t > 0$$

Proof II

- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies $\frac{f(k^*)}{k^*} = \frac{\delta}{s}$.

Non-Existence and Non-Uniqueness

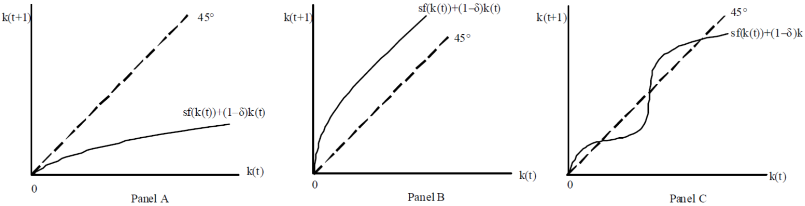


Figure: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Comparative statics with respect to s and δ are straightforward for k^* and y^*. s, δ 比较静态对 k*, y* 成立
- But c^* will not be monotone in the saving rate (think, for example, of s = 1). c* 在 s 中不单调
- In fact, there will exist a specific level of the saving rate, s_{gold} , referred to as the "golden rule" saving rate, which maximizes c^*.
- But cannot say whether the golden rule saving rate is "better" than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$c^*(s) = (1 - s)f(k^*(s)) = f(k^*(s)) - \delta k^*(s)$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

$c^*(s) = (1-s) f(k^*(s)) = f(k^*(s)) - \delta k^*(s)$

$c^*(s) = f(k^*(s)) - \delta k^*(s)$

- Differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta]\frac{\partial k^*(s)}{\partial s}$$

- s_{gold} is such that $\frac{\partial c^*(s_{gold})}{\partial s} = 0$. The corresponding steady-state golden rule capital stock is defined as k_{gold}.
  Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold} , with the corresponding steady state capital level k_{gold} such that

$$f'(k^*_{gold}(s)) = \delta$$

The Golden Rule

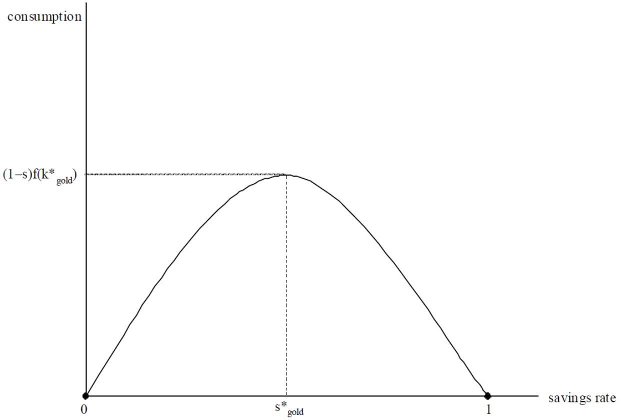


Figure: The "golden rule" level of savings rate, which maximizes steady-state consumption.

Dynamic Inefficiency

- When the economy is below k_{gold}, higher saving will increase consumption; when it is above k_{gold}, steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (dynamic inefficiency).
- But no utility function, so statements about "inefficiency" have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

Discrete-Time Solow Model Redux((艺术作品)以新方式呈现的)

- Per capita capital stock evolves according to

$$k_{t+1} = sf(k_t) + (1-\delta)k_t.$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by

$$C_t = (1-s)Y_t.$$

- And factor prices are given by

$$R_t = f'(k_t) > 0$$
$$w_t = f(k_t) - f'(k_t)k_t > 0$$

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus the steady state equilibrium.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the "transitional dynamics" of the equilibrium difference equation $k_{t+1} = (1-\delta)k_t + sf(k_t)$ starting from an arbitrary initial capital-labor ratio $k_0 > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics in the Discrete Time Solow Model

Proposition

Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation $k_{t+1} = (1-\delta)k_t + sf(k_t)$ is globally asymptotically stable, and starting from any $k_0 > 0$, k_t monotonically converges to k^*.

Proof of Proposition: Transitional Dyamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k) > 0$ for all k.
- Next from the equation $k_{t+1} = (1-\delta)k_t + sf(k_t)$,

$$k_{t+1} = g(k_t),$$

  with a unique steady state at k^*
- From $\frac{f(k^*)}{k^*} = \frac{\delta}{s}$, the steady state capital k^* satisfies $\delta k^* = sf(k^*)$, or

$$k^* = g(k^*)$$

- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) > 0$ from Assumption 2.

凹函数（Concave Function）

Concave function

25 languages

Article Talk

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From Wikipedia, the free encyclopedia

In mathematics, a **concave function** is the negative of a convex function. A concave function is also synonymously called **concave downwards**, **concave down**, **convex upwards**, **convex cap**, or **upper convex**.

Definition [edit]

A real-valued function f on an interval (or, more generally, a convex set in vector space) is said to be *concave* if, for any x and y in the interval and for any $\alpha \in [0,1]$,[1]

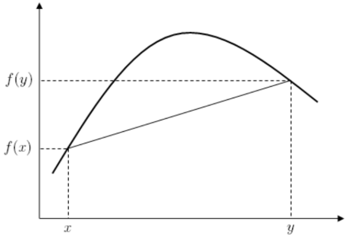
$$f((1-\alpha)x+\alpha y) \geq (1-\alpha)f(x)+\alpha f(y)$$

A function is called *strictly concave* if

$$f((1-\alpha)x+\alpha y) > (1-\alpha)f(x)+\alpha f(y)$$

for any $\alpha \in (0,1)$ and $x \neq y$.

For a function $f:\mathbb{R}\to\mathbb{R}$, this second definition merely states that for every z strictly between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$.



凸函数（Convex Function）

Definition [edit]

Let X be a convex subset of a real vector space and let $f : X \to \mathbb{R}$ be a function.

Then f is called ***convex*** if and only if any of the following equivalent conditions hold:

1. For all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$:

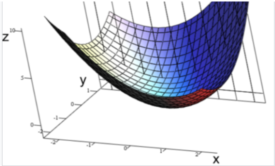
   $$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

   The right hand side represents the straight line between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in the graph of f as a function of t; increasing t from 0 to 1 or decreasing t from 1 to 0 sweeps this line. Similarly, the argument of the function f in the left hand side represents the straight line between x_1 and x_2 in X or the x-axis of the graph of f. So, this condition requires that the straight line between any pair of points on the curve of f to be above or just meets the graph.[2]

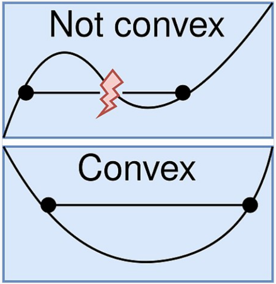
2. For all $0 < t < 1$ and all $x_1, x_2 \in X$ such that $x_1 \neq x_2$:

   $$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

   The difference of this second condition with respect to the first condition above is that this condition does not include the intersection points (for example, $(x_1, f(x_1))$ and $(x_2, f(x_2))$) between the straight line passing through a pair of points on the curve of f (the straight line is represented by the right hand side of this condition) and the curve of f; the first condition includes the intersection points as it becomes $f(x_1) \leq f(x_1)$ or $f(x_2) \leq f(x_2)$ at $t = 0$ or 1, or $x_1 = x_2$. In fact, the intersection points do not need to be considered



A graph of the bivariate convex function $x^2 + xy + y^2$.



Convex vs. Not convex

Proof of Proposition: Transitional Dyamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k),$$

- The second inequality uses the fact that $f(0) \geq 0$
- The above equation implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The simple result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dyamics III

- To prove global stability, note that for all $k_t \in (0, k^*)$,

$$k_{t+1} - k^* = g(k_t) - g(k^*) = -\int_{k_t}^{k^*} g'(k)dk < 0$$

- Second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k.

Proof of Proposition: Transitional Dyamics IV

- Next, the law of motion for per capita capital also implies

$$\frac{k_{t+1} - k_t}{k_t} = s\frac{f(k_t)}{k_t} - \delta > s\frac{f(k^*)}{k^*} - \delta = 0.$$

  Moreover, for any $k_t \in (0, k^* - \epsilon)$, this is uniformly so.
- Second line uses the fact that $f(k)/k$ is decreasing in k and last line uses the definition of k^*.
- These two arguments together establish that for all $k_t \in (0, k^*)$, $k_{t+1} \in (k_t, k^*)$.
- An identical argument implies that for all $k_t > k^*$, $k_{t+1} \in (k^*, k_t)$.
- Therefore, $\{k_t\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock $k_0 < k^*$, economy grows towards k^*, capital deepening and growth of per capita income.
  - If economy were to start with $k_0 > k^*$, reach the steady state by decumulating capital and contracting.
- As a consequence:
  Proposition Suppose that Assumptions 1 and 2 hold, and $k_0 < k^*$, then $\{w_t\}_{t=0}^{\infty}$ is an increasing sequence and $\{R_t\}_{t=0}^{\infty}$ is a decreasing sequence. If $k_0 > k^*$, the opposite results apply.
- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k_0 < k^*$.

Transitional Dynamics in Figure

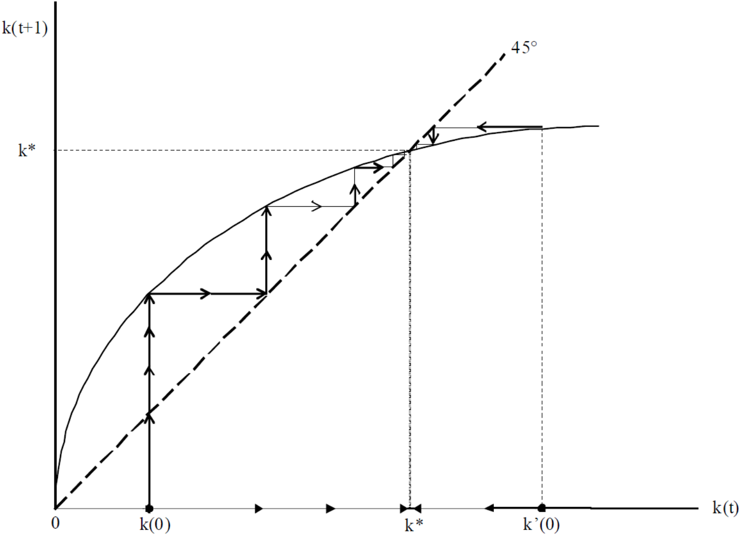


Figure: Transitional dynamics in the basic Solow model.

From Difference to Differential Equations I

- Start with a simple difference equation

$$x_{t+1} - x_t = g(x_t).$$

- Now consider the following approximation for any $\Delta t \in [0, 1]$,

$$x_{t+\Delta t} - x_t \approx \Delta t g(x_t).$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives the first equation on this slide.
- In-between it is a linear approximation, not too bad if

$$g(x) \approx g(x_t) \;\; for \;\; all \;\; x \in [x_t, x_{t+1}]$$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{\Delta t} = \dot{x}_t \approx g(x_t),$$

  where

$$\dot{x}_t \equiv \frac{dx_t}{dt}$$

- This equation is a differential equation representing similar equation from last slide for the case in which t and $t+1$ is "small".

The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, factor prices equations are as before, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S_t = sY_t.$$

- Consumption equation is as before.
- Introduce population growth,

$$L_t = exp(nt)L_0.$$

The Fundamental Equation of the Solow Model in Continuous Time II

- Recall

$$k_t \equiv \frac{K_t}{L_t}$$

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- Implies

$$\dot{k}_t = \frac{\dot{K}_t}{L_t} - \frac{\dot{L}_t}{L_t^2} K_t$$

- Divided by k_t:

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} - n$$

人均资本增速 = 总资本增速 − 人口增速

The Fundamental Equation of the Solow Model in Continuous Time III

- From previous slides defining instantaneous change in a variable:

$$\dot{K}_t = sF(K_t, L_t, A_t) - \delta K_t$$
$$\frac{\dot{K}_t}{L_t} = sf(k_t) - \delta \frac{K_t}{L_t}$$
$$\frac{\dot{K}_t}{K_t} = \frac{sf(k_t)}{k_t} - \delta$$

- Recall the last equation from previous slide:

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - n = \frac{sf(k_t)}{k_t} - (\delta + n)$$

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (\delta + n)$$

The Fundamental Equation of the Solow Model in Continuous Time IV

Definition In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K_t, L_t, Y_t, C_t, w_t, R_t]_{t=0}^{\infty}$ such that L_t satisfies $L_t = exp(nt)L_0$, $k_t \equiv \frac{K_t}{L_t}$ satisfies $\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n)$, Y_t is given by the aggregate production function, C_t is given by $C_t = (1 - s)Y_t$, and w_t and R_t are given by $R_t = f'(k_t)$ and $w_t = f(k_t) - f'(k_t)k_t$.

- As before, steady-state equilibrium involves k_t remaining constant at some level k^*.

Steady State of the Solow Model in Continuous Time

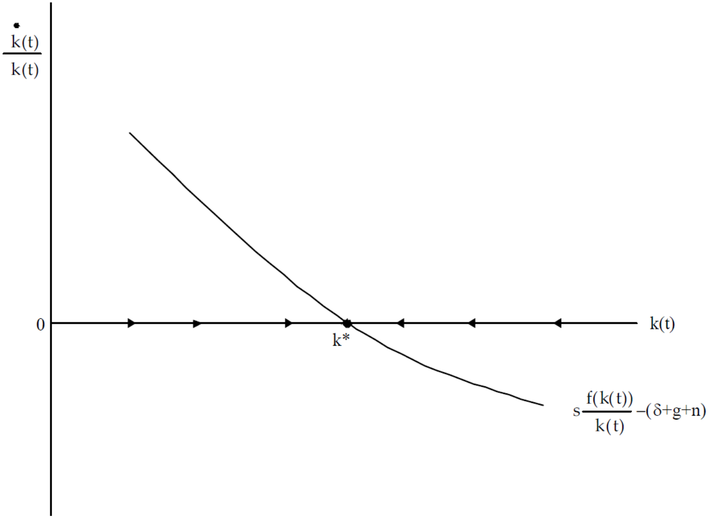
- Equilibrium path $\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n)$ has a unique steady state at k^*, which is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s}$$

  Proposition Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by the above equation, per capita output and per capita consumption are given accordingly.

- Similar comparative statics to the discrete time model.

Simple Result in Figure



Steady State With Population Growth

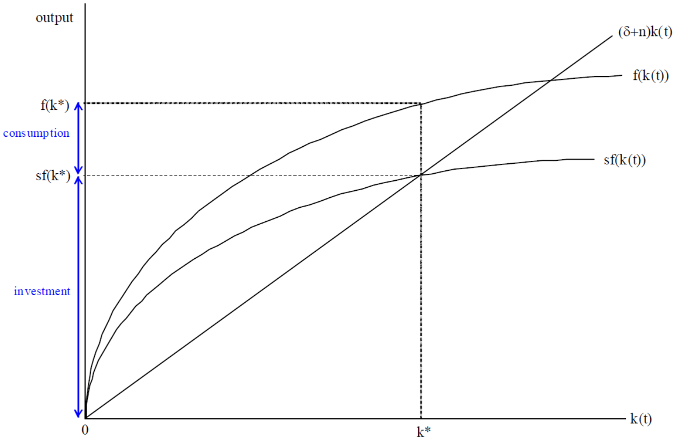


Figure: Investment and consumption in the steady-state equilibrium with population growth.

Transitional Dynamics in the Continuous Time Solow Model II

Proposition Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k_0 > 0$, $k_t \to k^*$.

- Proof: Follows immediately from the Theorem above by noting whenever $k < k^*$, $sf(k) - (n + \delta) > 0$ and whenever $k > k^*$, $sf(k) - (n + \delta) < 0$.

The Solow Growth Model with Technological Progress: Continuous Time I 劳动扩张型

- Production function must admit representation of the form

$$Y_t = F(K_t, A_t L_t)$$

- Moreover, suppose

$$\frac{\dot{A}_t}{A_t} = g,$$
$$\frac{\dot{L}_t}{L_t} = n$$

- Again using the constant saving rate

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define k_t as the effective capital-labor ratio, i.e.,

$$k_t \equiv \frac{K_t}{A_t L_t}.$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - n - g.$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}_t &\equiv \frac{Y_t}{A_t L_t} \\ &= F\left(\frac{K_t}{A_t L_t}, 1\right) \equiv f(k_t) \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is

$$y_t = A_t \hat{y}_t = A_t f(k_t).$$

- Clearly if $\hat{y}_t$ is constant, income per capita, y_t, will grow over time, since A_t is growing.
- Thus should not look for "steady states" where income per capita is constant, but for balanced growth paths, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}_t$ or k_t remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms "steady state" and balanced growth path interchangeably.
- Substituting for $\dot{k}_t$:

$$\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n + g).$$

- Only difference is the presence of g: k is no longer the capital-labor ratio but the effective capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition
Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as before. Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}.$$

Per capita output and consumption grow at the rate g.

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation $\frac{f(k^*)}{k^*} = \frac{\delta+g+n}{s}$, emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
  (1) depreciation at the rate δ.
  (2) population growth at the rate n, which reduces capital per worker.
  哈罗德中性
  (3) Harrod-neutral technological progress at the rate g.
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

The Solow Growth Model with Technological Progress: Continuous Time VII

Proposition
Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k_0 > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k_t \to k^*$).

- Now model generates growth in output per capita, but entirely exogenously.

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.

Comparative Dynamics in Figure

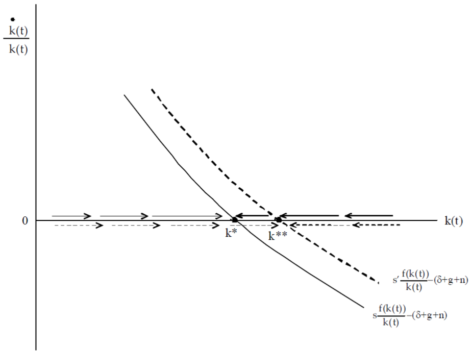


Figure: Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s'.
  - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}.
  - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.
  - Immediately, the capital stock remains unchanged (since it is a state variable).
  - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t$".
  - Starting at t', the economy follows the rightwards arrows until t'.
  - After t'', the original steady state of the differential equation applies and leftwards arrows become effective.
  - From t'' on wards, economy gradually returns back to its original balanced growth equilibrium, k^*.

Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.