Neoclassical Growth Model

1-36

不考 Transversality condition

No-Ponzi game condition

Ramsey Model (Cass-Koopmans Model)

- Solow model: constant saving rate. $\bar{s}$
- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and 内生的 s endogenizes savings. This model specifies the preference orderings of individuals and derives their decisions from these preferences. It also
  - Enables better understanding of the factors that affect savings decisions.
  - Enables to discuss the "optimality" of equilibria
  - Clarifies whether the (competitive) equilibria of growth models can be "improved upon".
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

Preliminaries 准备工作

- Consider an economy consisting of a unit measure of infinitely-lived households.
- I.e., an uncountable number of households: e.g., the set of
  标准化 H households H could be represented by the unit interval [0, 1].
- Emphasize that each household is infinitesimal and will have no effect on aggregates. 影响小
  same
- Can alternatively think of H as a countable set of the form $H = \{1, 2, \cdots, M\}$ with $M = \infty$, without any loss of generality.
- Advantage of unit measure: averages and aggregates are the same.

Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.

  时间可分的偏好
- In addition, time separable preferences: each household i has an instantaneous (Bernoulli) utility function (or felicity function): 瞬时的

$$u_i(c_{i,t}),$$

$u''(\cdot) < 0$

- $u_i : R_+ \to R$ is increasing and concave and $c_{i,t}$ is the consumption of household i. 凹的

Infinite Horizon and the Representative Household

- Given by the von Neumann-Morgenstern expected utility function

$$E_0^i \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}),$$

  where $\beta_i \in (0,1)$ is the discount factor of household i, where $T < \infty$ or $T = \infty$, corresponding to finite planning horizon (e.g., with overlapping generations) or infinite planning horizon.

- Exponential discounting and time separability also ensure "time-consistent" behavior.

一贯的

- A solution $\{x_t\}_{t=0}^{T}$ (possibly with $T = \infty$) is time consistent if: whenever $\{x_t\}_{t=0}^{T}$ is an optimal solution starting at time $t = 0$, $\{x_t\}_{t=t'}^{T}$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$.

t=0 规划的最优解，经过t'后仍为最优解.

Challenges to the Representative Household

- An economy admits a representative household if preference side can be represented as if a single household made the aggregate consumption and saving decisions subject to a single budget constraint.
- Simplest case that will lead to the existence of a representative household: suppose each household is identical.
- But we can do a little, but not much, better than that: aggregation problems and Gorman preferences (out of the scope of this course, but some intuition here).

Preferences, Technology and Demographics I

- Infinite-horizon, continuous time. 连续时间无穷期
- Representative household with instantaneous utility function 瞬时

$$u(c_t),$$

Assumption, $u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies the following Inada type assumptions: $u'>0,\ u''<0$

$$\lim_{c\to 0} u'(c) = \infty; \lim_{c\to\infty} u'(c) = 0$$

- Suppose representative household represents set of identical households (normalized to 1).
- Each household has an instantaneous utility function given by the above utility function.
- $L_0 = 1$ and

$L_t = e^{nt}$

$$L_t = exp(nt)$$

Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at $t = 0$:

时间偏好⇔β

$e^{-\rho} = \beta$

$$U_0 \equiv \int_0^{\infty} e^{-\rho t} L_t u(c_t) dt = \int_0^{\infty} e^{-(\rho - n)t} u(c_t) dt, \quad (1)$$

  where c_t consumption per capita at t, ρ is the subjective discount rate, and effective discount rate is $\rho - n$.
- Continues time analogue of $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})$
- Objective function (1) embeds
  利他
  (a) Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively. (b) strictly concave of $u(\cdot)$.
- Thus each household member will have an equal consumption

$$c_t \equiv \frac{C_t}{L_t}$$

Preferences, Technology and Demographics III

- Utility of $u(c_t)$ per household member at time t, total of $L_t u(c_t) = exp(nt)u(c_t)$.
- With discount at rate of $exp(-\rho t)$, obtain (1).
- **Assumption**

$$\rho > n$$

- Ensures that in the model without growth, discounted utility is finite (otherwise infinite utility and not well behaved equilibrium). Will strengthen it in model with growth.
- Start model without any technological progress.
- Factor and product markets are competitive.
- Production possibilities set of the economy is represented by:

$$Y_t = F(K_t, L_t), \tag{2}$$

- Standard constant returns to scale and Inada assumptions still hold.

Preferences, Technology and Demographics IV

- Per capita production function:

$$y_t \equiv \frac{Y_t}{L_t} = F[\frac{K_t}{L_t}, 1] = f(k_t),$$

where, as before, k_t is capital per capita.

- Competitive factor markets then imply:

$$\begin{aligned} R_t &= F_K[K_t, L_t] = f'(k_t); \\ w_t &= F_L[K_t, L_t] = f(k_t) - k_t f'(k_t). \end{aligned}$$

Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time t by $\mathcal{A}_t$, Then,

$$\text{资产变动}\quad \dot{\mathcal{A}}_t = r_t\mathcal{A}_t + w_tL_t - c_tL_t$$

  r_t is the risk-free market flow rate of return on assets, and w_tL_t is the flow of labor income earnings of the household.

- Defining per capita assets as:

$$a_t \equiv \frac{\mathcal{A}_t}{L_t}$$

  we obtain:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t.$$

- Household assets can consist of capital stock, K_t which they rent to firms and government bonds, B_t.

Preferences, Technology and Demographics VI

- With uncertainty, households would have a 投资组合 portfolio choice between K_t and riskless bonds.
- With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.
- (The total net supply of bonds is zero in a closed economy without government) Thus, market clearing:

$$a_t = k_t.$$ 人均资产=人均资本存量

- No uncertainty depreciation rate of δ implies

$$r_t = R_t - \delta.$$

资产无风险回报率=租金-折旧率.

The Budget Constraint

- The differential equation:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t. \tag{3}$$

  is a flow constraint 流量约束

- Not sufficient as a proper budget constraint unless we impose a lower bound on assets.
- Three options: (1) Lower bound on assets such as $a_t \geq 0$ for all t, (2) Natural debt limit. (3) No Ponzi Game Condition.
- Let us focus on the latter.

The No Ponzi Game Condition

有限时间：$\begin{cases} \text{横断} & a_t \le 0 \\ \text{Ponzi} & a_t \geq 0 \end{cases}$

- Infinite-horizon no Ponzi game condition is:

折现

$$\lim_{t\to\infty} a_t \boxed{e^{-\int_0^t (r_s - n)ds}} \geq 0$$

a_∞的现值 ≥ 0.

- Transversality condition ensures individual would never want to have positive wealth asymptotically, 渐近地 so no Ponzi game condition can be strengthened to (though not necessary in general):

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$$

Understanding the No Ponzi Game Condition [*]

- Why?
- Write the single budget constraint of the form:

$$\int_0^T c_t L_t exp(\int_t^T r_s ds)dt + A_T =$$

$$\int_0^T w_t L_t exp(\int_t^T r_s ds)dt + A_0 exp(\int_0^T r_s ds)$$

- Differentiating with respect to T and dividing L_t gives the asset evolving equation (budget constraint, Equation 3).
- Now imagine that the above equation applies to a finite-horizon economy.
- Flow budget constraint (asset) does not gurantee that $A_T \geq 0$.
- Thus in finite-horizon we would simply impose the above equation as a boundary condition.
- The no Ponzi game condition is the infinite horizon equivalent of this (obtained by dividing by L_t and multiplying both sides by $exp(\int_t^T r_s ds)$ and taking the limit as $T \to \infty$).

Definition of Equilibrium (*)

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c_t, k_t, w_t, R_t]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock K_0, and the time path of prices $[w_t, R_t]_{t=0}^{\infty}$, and all markets clear.
Definition A competitive equilibrium of the Ramsey economy consists of paths $[c_t, k_t, w_t, R_t]_{t=0}^{\infty}$, such that the representative household maximizes $U_0 = \int_0^{\infty} e^{-(\rho-n)t} u(c_t) dt$, subject to $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ and $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0$ given initial capital-labor ratio k_0. Factor prices are given by $R_t = F_K[K_t, L_t] = f'(k_t)$ and $w_t = F_L[K_t, L_t] = f(k_t) - k_t f'(k_t)$. and the rate of return on assets given by $r_t = R_t - \delta$.

Household Maximization I (as given)

a、状态变量

c：自由选择，控制变量

- set up the current-value Hamiltonian:

$$H(a, c, \mu) = u(c_t) + \mu_t[(r_t - n)a_t + w_t - c_t]$$

- Maximum Principle → "candidate solution"

$$H_c(a, c, \mu) = u'(c_t) - \mu_t = 0 \tag{4}$$

Continuous Time Euler Equ.

$$H_a(a, c, \mu) = \mu_t(r_t - n) = -\dot{\mu}_t + (\rho - n)\mu_t \tag{5}$$

NoPonzi & 横断

$$\lim_{t\to\infty} [\underbrace{e^{-(\rho-n)t}}_{\text{实际折现（折现-人口增长）}} \mu_t a_t] = 0 \tag{6}$$

$\mu_t = u'(c_t) \neq 0.$

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t. \tag{7}$$

流动＋两端限制 ⇒ Buget constraint

- Equation 5 from page 289 (derivation of these is out of the scope of this class, some intuition here).
- Notice transversality condition is written in terms of the current-value costate variable.

(5) 相当于连续时间的 Euler Equation、

$$H_a(a,c,\mu) = \mu_t(r_t - n) = -\dot{\mu}_t + \underbrace{(\rho - n)}_{\text{discount rate (effective)}}\mu_t .$$

Discrete Euler: $u'(c_t) = \beta(1+r_t)u'(c_{t+1})$

$$\Delta t \to 0. \quad \frac{(\mu_{t+\Delta t} - \mu_t)}{\Delta t} = (\rho - n)\mu_{t+\Delta t}(1 + r\Delta t)$$

$$\dot{\mu}_t = (\rho - n)\mu_t$$

Household Maximization II

- For any $\mu_t > 0$, $H_c(a, c, \mu)$ is a concave function of (a, c) and strictly concave in c.
- The first necessary condition implies $\mu_t > 0$ for all t.
- Therefore, sufficient conditions imply that the candidate solution is an optimum.
- Rearrange the second condition:

(5) $\mu_t(r_t - n) = -\dot{\mu}_t + (\rho - n)\mu_t$

$r_t - n = -\frac{\dot{\mu}_t}{\mu_t} + (\rho - n)$

$$\frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho) \quad \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho) \tag{8}$$

$r_t > \rho$, $\dot{\mu}_t < 0$, $c\uparrow$

- First necessary condition implies that:

$$u'(c_t) = \mu_t$$

- Differentiate with respect to time and divide by μ_t,

① 对时间求导

$\frac{c_t \cdot u''(c_t)\dot{c}_t}{u'(c_t) = \mu(t) \cdot c_t} = \frac{\dot{\mu}_t}{\mu_t}$

② 配凑

$$-\epsilon \triangleq \boxed{\frac{u''(c_t)c_t}{u'(c_t)}} \frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t} \tag{9}$$

Household Maximization III

- Plug the above equation (Equation 9) into Equation 8:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(r_t - \rho)$$

where

弹性 $e = \frac{\Delta Q/\Delta P}{Q/P}$ $\hat{Q} = u'(c_t)$ $\hat{P} = c_t$.

$\epsilon(c_t) = \frac{u''(c_t)}{u'(c_t)/c_t} = \frac{u''(c_t) \cdot c_t}{u'(c_t)}$

$$\epsilon(c_t) \triangleq -\frac{u''(c_t)c_t}{u'(c_t)}$$

is the elasticity of the marginal utility $u'(c_t)$. 消费边际效用的弹性.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.
- Speed at which consumption will grow is related to the elasticity of marginal utility of consumption, $\epsilon(c_t)$.
- Even more importantly, $\epsilon(c_t)$ is the inverse of the intertemporal elasticity of substitution (see this more clearly in a later example). 跨期替代弹性.

Household Maximization IV

$$\frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho) \quad (8)$$

$$\int_0^t \frac{\dot{\mu}_t}{\mu_t} = -\int_0^t (r_s - \rho) ds,$$

- Integrating Equation 8,

$$\Rightarrow \ln \mu_t \Big|_0^t = -\int_0^t (r_s - \rho) ds \Rightarrow$$

$$\mu_t = \mu_0 e^{-\int_0^t (r_s - \rho) ds} = u'(c_0) e^{-\int_0^t (r_s - \rho) ds}$$

- Substituting into transversality condition (Equation 6),

$$0 = \lim_{t \to \infty} [e^{-(\rho - n)t} a_t u'(c_0) e^{-\int_0^t (r_s - \rho) ds}]$$

$$\lim_{t \to \infty} [e^{-(\rho - n)t} a_t \mu_t] = 0$$ 代入

if and only if

$$0 = \lim_{t \to \infty} [a_t e^{-\int_0^t (r_s - n) ds}]$$

- Thus the "strong version" of the no-Ponzi condition.

Household Maximization V

- Since $a_t = k_t$, transversality condition is also equivalent to:

$$\lim_{t\to\infty}[e^{-\int_0^t (r_s - n)ds} k_t] = 0$$

- Notice term $e^{-\int_0^t r_s ds}$ is a present-value factor: converts a unit of income at t to a unit of income at 0;
- When $r_s = r$, factor would be e^{-rt}. More generally, define an average interest rate between dates 0 and t.

$$\bar{r_t} = \frac{1}{t}\int_0^t r_s ds$$

- Thus conversion factor between dates 0 and t is $e^{-\bar{r}_t t}$. And the transversality condition

$$\lim_{t\to\infty}[e^{-(\bar{r_t} - n)t} a_t] = 0$$

Equilibrium Prices

- Equilibrium prices given by equations about factor prices；
- Thus market rate of return for consumers,

$$r_t = f'(k_t) - \delta$$ 要素市场均衡

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(f'(k_t) - \delta - \rho)$$

Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints: 可行性

$$\max_{[k_t, c_t]_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho - n)t} u(c_t) dt,$$

subject to

$$\dot{k_t} = f(k_t) - (n + \delta)k_t - c_t,$$

and $k_0 > 0$

Optimal Growth II

- Again set up the current-value Hamiltonian:

$$H(k,c,\mu) = u(c_t) + \mu_t[f(k_t) - (n+\delta)k_t - c_t]$$

- Candidate solution from the Maximum Principle:

$$\begin{aligned} H_c(k,c,\mu) &= u'(c_t) - \mu_t = 0, \\ H_k(k,c,\mu) &= \mu_t(f'(k_t) - \delta - n) = -\dot{\mu}_t + (\rho - n)\mu_t, \end{aligned}$$
$$\lim_{t\to\infty}[e^{-(\rho-n)t}\mu_t k_t] = 0$$

- 汉密尔顿函数在k上是严格凹的（strictly concave）。由充分性定理知道，这是唯一解。

Optimal Growth III

$$\begin{cases} r_t = f'(k_t) - \delta \\ \frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(r_t - \rho) \end{cases}$$

- Repeating the same steps as before, these imply:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(f'(k_t) - \delta - \rho) \tag{10}$$

and the transversality condition:

$$\lim_{t \to \infty} [k_t e^{-\int_0^t (f'(k_s) - \delta - n) ds}] = 0$$

Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{k}_t = 0 \;\; and \;\; \dot{c}_t = 0$$

$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta - \rho) = 0$

故 $f'(k_t) = \delta + \rho$.

- As long as $f(k^*) > 0$, irrespective of the exact utility function, we must have a capital-labor ratio k^* such that

$f'(k^*) > f'(k^*)_{Solow}$

$$f'(k^*) = \rho + \delta$$

此时 $k^* < k^*_{Solow}$.

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- Modified golden rule: level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.

这里家庭有时间概念. 未来consumption打折.

Steady-State Equilibrium II

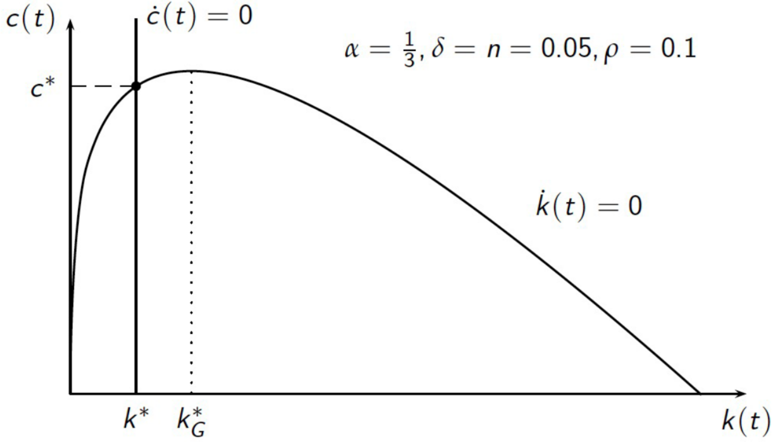


Figure: Steady state in the baseline neoclassical growth model

Steady-State Equilibrium III

- Given k^*, steady-state consumption level:

$$? \quad c^* = f(k^*) - (n + \delta)k^*. \quad \text{same}$$

- A steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Steady-State Equilibrium IV

- Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as

$$? \quad s^* = \frac{(n+\delta)k^*}{f(k^*)} \qquad \frac{f(k^*)}{k^*} = \frac{\delta}{s}$$

- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
  - result depends on the way in which intertemporal discounting takes place.
- k^* and thus c^* do not depend on the instantaneous utility function $u(\cdot)$.(1.form of the utility function only affects the transitional dynamics; 2.not true when there is technological change.)

Transitional Dynamics I

$\dot{k}_t = 0 \Rightarrow c_t = f(k_t) - (n+\delta) k_t$

$\dot{c}_t = 0 \Rightarrow f'(k_t) = \delta + \rho$

- Equilibrium is determined by two differential equations:

$$\dot{k_t} = f(k_t) - (n+\delta)k_t - c_t$$
$$\frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(f'(k_t) - \delta - \rho)$$

- Moreover, we have an initial condition k_0, also a boundary condition at infinity,

$$\lim_{t\to\infty}[k_t e^{-\int_0^t (f'(k_s)-\delta-n)ds}] = 0$$

Transitional Dynamics III

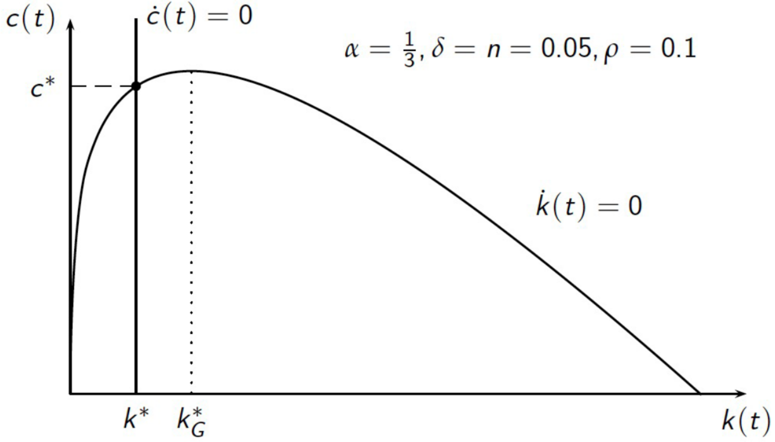
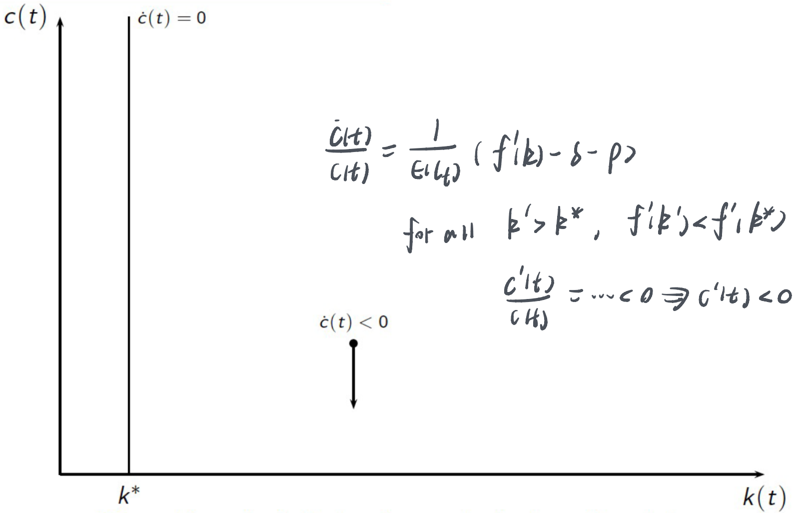


Figure: Steady state in the baseline neoclassical growth model

过渡动态



$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon(c(t))}\left(f'(k) - \delta - \rho\right)$$

for all $k' > k^*$, $f'(k') < f'(k^*)$

$$\frac{\dot{c}(t)}{c(t)} = \cdots < 0 \Rightarrow \dot{c}(t) < 0$$

Figure: Dynamics in the baseline neoclassical growth model

过渡动态

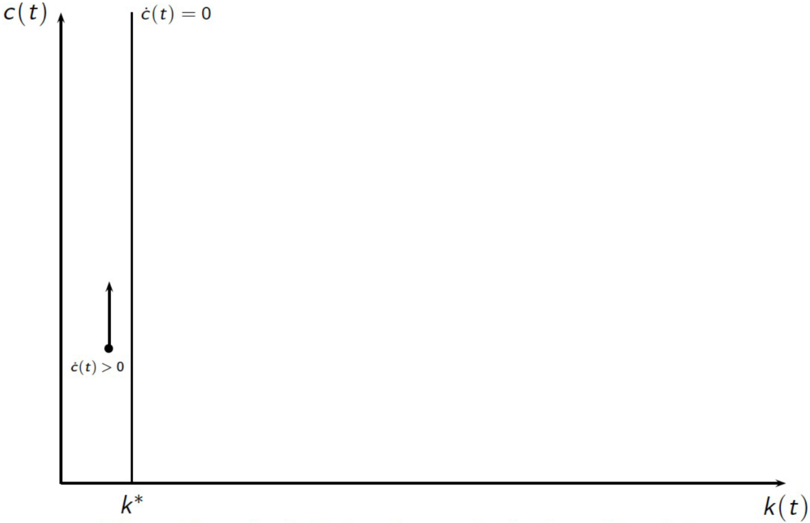
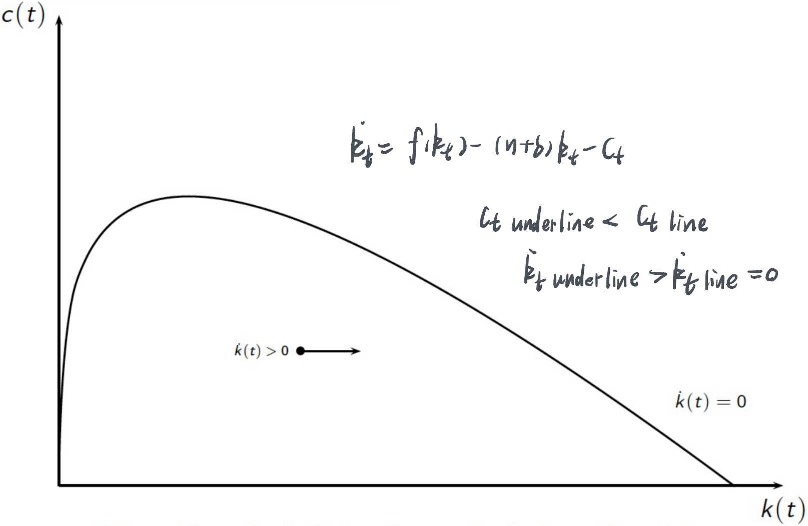


Figure: Dynamics in the baseline neoclassical growth model

过渡动态



$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t$

c_t underline < c_t line

$\dot{k}_t$ underline > $\dot{k}_t$ line = 0

Figure: Dynamics in the baseline neoclassical growth model

过渡动态

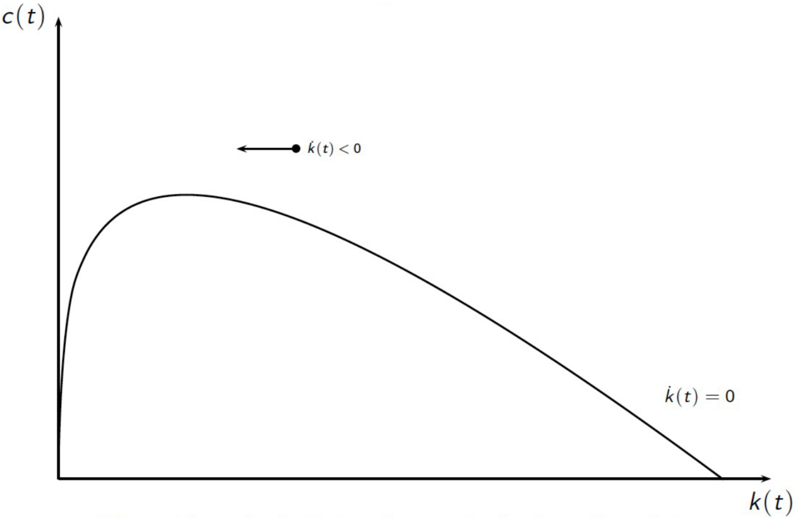


Figure: Dynamics in the baseline neoclassical growth model

过渡动态

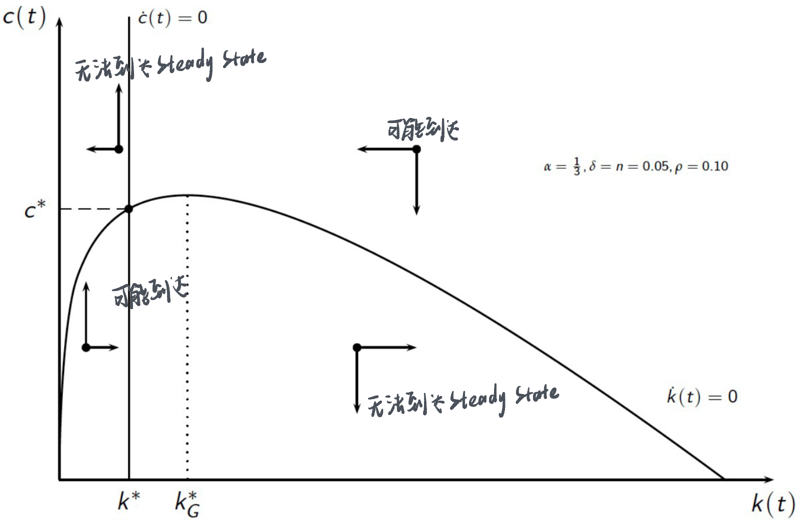


Figure: Dynamics in the baseline neoclassical growth model

过渡动态 (*)

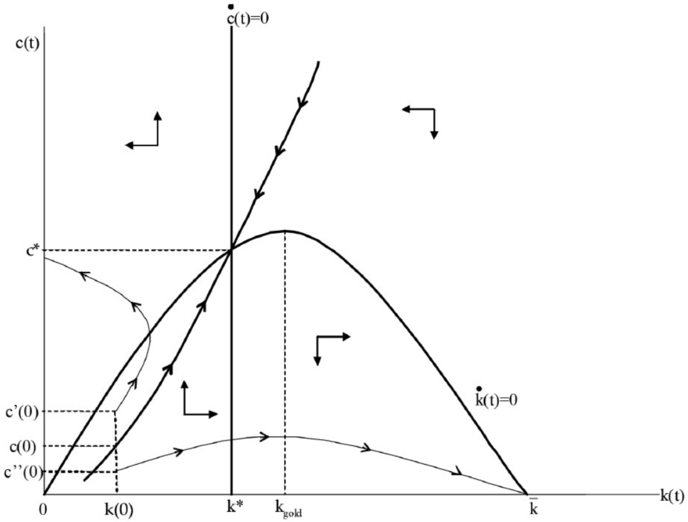


Figure: Transitional dynamics in the baseline neoclassical growth model

技术变动：technological change

技术变动：technological change

Technological Change and the Neoclassical Model

- Extend the production function to:

$$Y_t = F[K_t, A_tL_t] \tag{11}$$

where

$$A_t = exp(gt)A_0$$

- A consequence of Uzawa Theorem.: Equation 11 imposes purely labor-augmenting-Harrod-neutral- technological change.

Technological Change and the Neoclassical Model

- Define

$$\hat{y}_t \equiv \frac{Y_t}{A_t L_t} = F[\frac{K_t}{A_t L_t}, 1] \equiv f(k_t)$$

where

$$k_t \equiv \frac{K_t}{A_t L_t}$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

Technological Change III

- Define balanced growth as a pattern of growth consistent with the Kaldor facts of constant capital-output ratio and capital share in national income.
- These two observations together also imply that the rental rate of return on capital, R_t, has to be constant, which, from $r_t = R_t - \delta$, implies that r_t has to be constant.
- Again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP).

$$\boxed{\frac{\dot{c}_t}{c_t} = \frac{1}{\epsilon(c_t)}(r_t - \rho)}$$

- If $r_t \to r^*$, then $\frac{\dot{c}_t}{c_t} \to g_c$ is only possible if $\epsilon(c_t) \to \epsilon$. (The elasticity of marginal utility of consumption is asymptotically constant.) 渐近恒定

Technological Change IV (*)

- Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with:

$$u(c) = \begin{cases} \frac{1}{1-\theta} c_t^{1-\theta} & \theta > 0 \; and \; \theta \neq 1 \\ lnc_t & \theta = 1 \end{cases}$$

- Elasticity of marginal utility of consumption, ϵ, is given by θ.
- When $\theta = 0$, these represent linear preferences, when $\theta = 1$, we have log preferences, and as $\theta \to \infty$, infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences:

$$\int_0^\infty e^{-(\rho-n)t} \frac{\tilde{c_t}^{1-\theta} - 1}{1-\theta} dt \qquad (12)$$

Technological Change V (*)

- $\tilde{c}_t \equiv \frac{C_t}{L_t}$ is per capita consumption.
- Refer to this model, with labor-augmenting technological change and CRRA preference as given by Equation 12 as the canonical model.
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(r_t - \rho)$$

- Steady state equilibrium first: since with technological progress there will be growth in per capita income, $\tilde{c}_t$ will grow.

Technological Change VI

- Instead define

$$c_t \equiv \frac{C_t}{A_t L_t} \equiv \frac{\tilde{c}_t}{A_t}$$

- This normalized consumption level will remain constant along the BGP:

效率人均 人均

$$\boxed{\frac{\dot{c}_t}{c_t} \equiv \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} - g = \frac{1}{\theta}(r_t - \rho - \theta g)}$$

- For the accumulation of capital stock:

$$\dot{k}_t = f(k_t) - c_t - (n + g + \delta)k_t$$

where $k_t \equiv \frac{K_t}{A_t L_t}$.

- Transversality condition, in turn, can be expressed as

$$\lim_{t \to \infty} \left\{ k_t e^{-\int_0^t [f'(k_s) - g - \delta - n] ds} \right\} = 0. \tag{13}$$

- In addition, equilibrium r_t is similarly determined as before, so

$$r_t = f'(k_t) - \delta$$

Technological Change VII

- Since in steady state c_t must remain constant:

$$r_t = \rho + \theta g$$

or

新增θ：替代弹性

$$f'(k^*) = \rho + \delta + \theta g \tag{14}$$

Pins down the steady-state value of the normalized capital ratio k^*

-

$$c^* = f(k^*) - (n + g + \delta)k^*$$

- Per capita consumption grows at the rate g.

Technological Change VIII

- Because there is growth, to make sure that the transversality condition is in fact satisfied substitute.
- Plug Equation 14 into Equation 13:

$$\lim_{t\to\infty}\{k_t e^{-\int_0^t[\rho-(1-\theta)g-n]ds}\}=0.$$

can only hold if $\rho-(1-\theta)g-n>0$, or alternatively:

$$\rho-n>(1-\theta)g.$$

- recall in steady state $r=\rho+\theta g$ and the growth rate of output is $g+n$.Therefore, equivalent to requiring that $r>g+n$.
- Steady-state capital-labor ratio no longer independent of preferences, depends on θ.

Technological Change IX

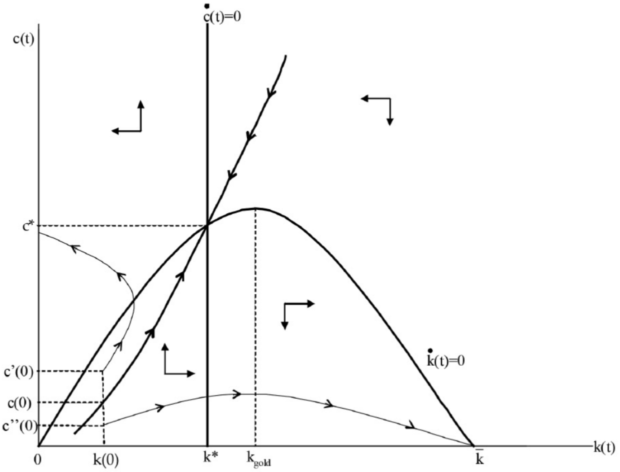


Figure: Transitional dynamics in the neoclassical growth model with technological change.

Comparative dynamics 比较动态

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- Look at the effect of a change in tax on capital (or discount rate ρ)
- Consider the neoclassical growth in steady state (k^*, c^*)
- Tax declines to $\tau' < \tau$
- Let the new steady state equilibrium be (k^{**}, c^{**})
- Since $\tau' < \tau$, $k^{**} > k^*$ while the equilibrium growth rate will remain unchanged.

The role of policy

- Example of effect of differences in policies;
- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{k}_t = f(k_t) - c_t - (n + g + \delta)k_t$$

- But interest rate faced by households changes to

$$r_t = (1 - \tau)(f'(k_t) - \delta)$$

The role of policy 2

- Growth rate of normalized consumption is then obtained from the consumer Euler equation,

$$
\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}(r_t - \rho - \theta g) \\
&= \frac{1}{\theta}(\underline{(1-\tau)}\underline{(f'(k_t) - \delta)} - \rho - \theta g)
\end{aligned}
$$

征税(资产税) 资产净回报率.

- Identical argument to that before implies:

$$
f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau}
$$

- Higher τ, since $f'(\cdot)$ is decreasing, reduces k^*
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
- But have not so far offered a reason why some countries may tax capital at a higher rate than others.

Conclusions

家庭最大化问题

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.
- This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.