第一点五章、基础知识

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基础知识准备

- 凹函数（Concave Function）
- 恒定(常数)替代弹性(CES)函数
- 恒定相对风险(CRRA)效用函数
- 包络定理（Envelope Theorem）
- 动态规划的例子 (Dynamic Programming: discrete time)
- 最优控制的例子 (Optimal Control: continous time)
- 消费决策（Consumption）

凹函数（Concave Function）

Concave function

文A 25 languages

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In mathematics, a **concave function** is the negative of a convex function. A concave function is also synonymously called **concave downwards**, **concave down**, **convex upwards**, **convex cap**, or **upper convex**.

Definition [edit]

A real-valued function f on an interval (or, more generally, a convex set in vector space) is said to be *concave* if, for any x and y in the interval and for any $\alpha \in [0,1]$,[1]

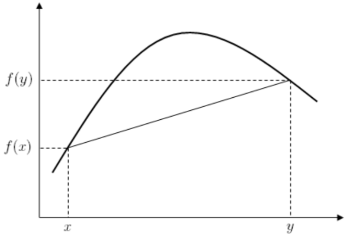
$$f((1-\alpha)x+\alpha y) \geq (1-\alpha)f(x)+\alpha f(y)$$

A function is called *strictly concave* if

$$f((1-\alpha)x+\alpha y) > (1-\alpha)f(x)+\alpha f(y)$$

for any $\alpha \in (0,1)$ and $x \neq y$.

For a function $f: \mathbb{R} \to \mathbb{R}$, this second definition merely states that for every z strictly between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$.



凸函数（Convex Function）

Definition [edit]

Let X be a convex subset of a real vector space and let $f : X \to \mathbb{R}$ be a function.

Then f is called ***convex*** if and only if any of the following equivalent conditions hold:

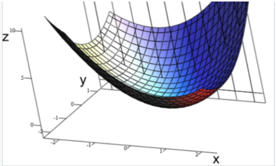
1. For all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$:

   $$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

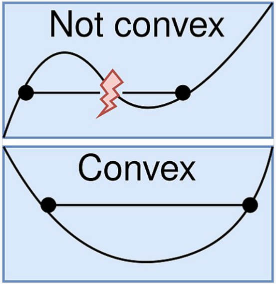
   The right hand side represents the straight line between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in the graph of f as a function of t; increasing t from 0 to 1 or decreasing t from 1 to 0 sweeps this line. Similarly, the argument of the function f in the left hand side represents the straight line between x_1 and x_2 in X or the x-axis of the graph of f. So, this condition requires that the straight line between any pair of points on the curve of f to be above or just meets the graph.[2]
2. For all $0 < t < 1$ and all $x_1, x_2 \in X$ such that $x_1 \neq x_2$:

   $$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

   The difference of this second condition with respect to the first condition above is that this condition does not include the intersection points (for example, $(x_1, f(x_1))$ and $(x_2, f(x_2))$) between the straight line passing through a pair of points on the curve of f (the straight line is represented by the right hand side of this condition) and the curve of f; the first condition includes the intersection points as it becomes $f(x_1) \leq f(x_1)$ or $f(x_2) \leq f(x_2)$ at $t = 0$ or 1, or $x_1 = x_2$. In fact, the intersection points do not need to be considered



A graph of the bivariate convex function $x^2 + xy + y^2$.



Convex vs. Not convex

恒定替代弹性函数: Constant Elasticity of Substitution Function

- 此函数既被用在偏好上，也被用在生产上
- 两个物品和多个物品的形式：

$$U = (a_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + a_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

σ：相互替代性

$$U = (\sum_{i=1}^{N} a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

a_i：权重，$a_i\uparrow$，x_i比重上升

恒定替代弹性函数: Constant Elasticity of Substitution Function

The CES production function is a neoclassical production function that displays constant elasticity of substitution. In other words, the production technology has a constant percentage change in factor (e.g. labour and capital) proportions due to a percentage change in marginal rate of technical substitution. The two factor (capital, labor) CES production function introduced by Solow,[2] and later made popular by Arrow, Chenery, Minhas, and Solow is:[3][4][5][6]

$$Q = F \cdot (a \cdot K^{\rho} + (1-a) \cdot L^{\rho})^{\frac{v}{\rho}}$$

where

- Q = Quantity of output
- F = Factor productivity
- a = Share parameter
- K, L = Quantities of primary production factors (Capital and Labor)
- $\rho = \frac{\sigma - 1}{\sigma}$ = Substitution parameter
- $\sigma = \frac{1}{1-\rho}$ = Elasticity of substitution
- v = degree of homogeneity of the production function. Where $v = 1$ **(Constant return to scale)**, $v < 1$ **(Decreasing return to scale)**, $v > 1$ **(Increasing return to scale)**.

恒定替代弹性函数: Constant Elasticity of Substitution Function

As its name suggests, the CES production function exhibits constant elasticity of substitution between capital and labor. Leontief, linear and Cobb–Douglas functions are special cases of the CES production function. That is,

- If ρ approaches 1, we have a linear or perfect substitutes function;
- If ρ approaches zero in the limit, we get the Cobb–Douglas production function;
- If ρ approaches negative infinity we get the Leontief or perfect complements production function.

$\rho \to 1$, $Q = K + \beta L$ 完全替代

$\rho \to 0$, $K^{r} L^{1-r}$

$\rho \to -\infty$, $Q = \min\{K, \beta L\}$. 完全互补

恒定替代弹性函数: Constant Elasticity of Substitution Function

The general form of the CES production function, with n inputs, is:[7]

$$Q = F \cdot \left[\sum_{i=1}^{n} a_i X_i^r \right]^{\frac{1}{r}}$$

where

- Q = Quantity of output
- F = Factor productivity
- a_i = Share parameter of input i, $\sum_{i=1}^{n} a_i = 1$
- X_i = Quantities of factors of production (i = 1,2...n)
- $s = \frac{1}{1-r}$ = Elasticity of substitution.

恒定替代弹性函数: Constant Elasticity of Substitution Function

> Assignment: why we say this is constant elasticity of substitution. Or show that the elasticity of substitution is constant at ρ.

Elasticity of substitution

10 languages

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Elasticity of substitution is the ratio of percentage change in capital-labour ratio with the percentage change in Marginal Rate of Technical Substitution.[1] In a competitive market, it measures the percentage change in the two inputs used in response to a percentage change in their prices.[2] It gives a measure of the curvature of an isoquant, and thus, the substitutability between inputs (or goods), i.e. how easy it is to substitute one input (or good) for the other.[3]

History of the concept [edit]

John Hicks introduced the concept in 1932. Joan Robinson independently discovered it in 1933 using a mathematical formulation that was equivalent to Hicks's, though that was not implemented at the time.[4]

Definition [edit]

The general definition of the elasticity of X with respect to Y is $E_Y^X = \frac{\%\text{ change in } X}{\%\text{ change in } Y}$, which reduces to $E_Y^X = \frac{dX}{dY}\frac{Y}{X}$ for infinitesimal changes and differentiable variables. The elasticity of substitution is the change in the ratio of the use of two goods with respect to the ratio of their marginal values or prices. The most common application is to the ratio of capital (K) and labor (L) used with respect to the ratio of their marginal products MP_K and MP_L or of the rental price (r) and the wage (w). Another application is to the ratio of consumption goods 1 and 2 with respect to the ratio of their marginal utilities or their prices. We will start with the consumption application.

Let the utility over consumption be given by $U(c_1, c_2)$ and let $U_{c_i} = \partial U(c_1, c_2)/\partial c_i$. Then the elasticity of substitution is:

$$E_{21} = \frac{d\ln(c_2/c_1)}{d\ln(MRS_{12})} = \frac{d\ln(c_2/c_1)}{d\ln(U_{c_1}/U_{c_2})} = \frac{\frac{d(c_2/c_1)}{c_2/c_1}}{\frac{d(U_{c_1}/U_{c_2})}{U_{c_1}/U_{c_2}}} = \frac{\frac{d(c_2/c_1)}{c_2/c_1}}{\frac{d(p_1/p_2)}{p_1/p_2}}$$

where MRS is the marginal rate of substitution. (These differentials are taken along the isoquant that passes through the base point. That is,

恒定相对风险效用函数: Constant Relative Risk Aversion Utility

- 对待风险的态度:risk averse， risk neutral， risk loving
- Risk aversion measurement 1:
  absolute risk aversion (ARA)

  c: 消费

$$A(c) = -\frac{u''(c)}{u'(c)}$$

- Risk aversion measurement 2:
  Relative risk aversion (RRA)

$$A(c) = -\frac{cu''(c)}{u'(c)}$$ 根据消费调整

恒定相对风险效用函数: Constant Relative Risk Aversion Utility

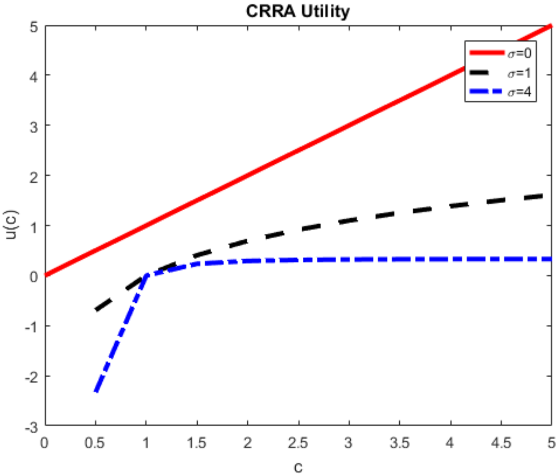
$u'(c) = c^{-\sigma}$ $u''(c) = -\sigma c^{-\sigma-1}$

$$u(c) = \begin{cases} \frac{1}{1-\sigma} c^{1-\sigma} & \sigma > 0 \; and \; \sigma \neq 1 \\ lnc & \sigma = 1 \end{cases}$$

$\varepsilon(c) = -\frac{u''(c)\cdot c}{u'(c)}$

$= \frac{\sigma c^{-\sigma-1}\cdot c}{c^{-\sigma}} = \sigma$

$\sigma \to 1$, $u(c) \to \ln c$.



恒定相对风险效用函数: Constant Relative Risk Aversion Utility

Suppose two goods:

$$U = u(c_1) + u(c_2)$$

The first derivative of the CRRA utility:

$$u'(c) = c^{-\sigma}$$

the marginal rate of substitution is:

$$MRS = \frac{u'(c_1)}{u'(c_2)} = \frac{c_1^{-\sigma}}{c_2^{-\sigma}} = (\frac{c_2}{c_1})^{\sigma}$$

$$\frac{c_2}{c_1} = (\frac{u'(c_1)}{u'(c_2)})^{\frac{1}{\sigma}} = (\frac{p_1}{p_2})^{\frac{1}{\sigma}}$$

$$\ln\frac{c_2}{c_1} = \frac{1}{\sigma}\ln\frac{p_1}{p_2}$$

恒定相对风险效用函数: Constant Relative Risk Aversion Utility

$$\frac{c_2}{c_1} = (\frac{p_1}{p_2})^{\frac{1}{\sigma}}$$

$\frac{1}{\sigma}$ measures the strength of substitution effect induced by the relative prices change.
The ratio between the consumption of the two goods depends only on the relative price, not the income level.
Consumption of both goods will go up in the same proportion as income (this is important: the productivity shock will bring more wealth. However, the ratio of current consumption to future consumption only depends on the relative price, NOT the wealth level.).

包络定理（Envelope Theorem）1

Let $f(x,a)$ be a C^1 function of $x \in R^n$ and the scalar a. For each choice of the parameter a, consider the unconstrained maximization problem:
max $f(x,a)$ with respect to x.
Let $x^*(a)$ be a solution of this problem. Suppose that $x^*(a)$ is a C^1 function of a. Then

$$\frac{df(x^*(a);a)}{da} = \frac{\partial f(x^*(a);a)}{\partial a}$$

包络定理（Envelope Theorem）2

This part is for constrained optimization problems.

$$M(a) = \max_{x_1,x_2} g(x_1, x_2, a)$$
$$st.h(x_1, x_2, a) = 0$$

The Lagrangian for this problem is

$$L = g(x_1, x_2, a) - \lambda h(x_1, x_2, a)$$

and the first-order conditions are

$$\begin{aligned}
\frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0 \\
\frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0 \\
h(x_1, x_2, a) &= 0
\end{aligned}$$

包络定理（Envelope Theorem）3

These conditions determine the optimal choice functions $(x_1(a), x_2(a))$ which in turn determine the maximum value function:

$$M(a) \equiv g(x_1(a), x_2(a), a)$$

The envelope theorem gives us a formula for the derivative of the value function with respect to a parameter in the maximization problem. Specifically the formula is

$$\begin{aligned}
\frac{dM(a)}{da} &= \frac{\partial L(x,a)}{\partial a}|_{x=x(a)} \\
&= \frac{\partial g(x_1, x_2, a)}{\partial a}|_{x=x(a)} - \lambda \frac{\partial h(x_1, x_2, a)}{\partial a}|_{x=x(a)}
\end{aligned}$$

As before, the interpretation of the partial derivatives needs special care: they are the derivatives of g and h with respect to a holding x_1 and x_2 fixed at their optimal values.

包络定理（Envelope Theorem）4

Proof:

$$\begin{aligned}\frac{dM}{da} &= \frac{\partial g}{\partial x_1}\frac{dx_1}{da} + \frac{\partial g}{\partial x_2}\frac{dx_2}{da} + \frac{\partial g}{\partial a} \\ &= \lambda[\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + \frac{\partial h}{\partial x_2}\frac{dx_2}{da}] + \frac{\partial g}{\partial a}\end{aligned}$$

Since $h(x_1, x_2, a) = 0$, we have

$$\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + \frac{\partial h}{\partial x_2}\frac{dx_2}{da} + \frac{\partial h}{\partial a} \equiv 0$$

plug back into the previous equation, we have the theorem.

The Deterministic Growth Model

$$
\begin{aligned}
V^*(k_0) &= \max_{k_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. \qquad k_{t+1} &= (1-\delta)k_t + i_t \\
y_t &= f(k_t) \\
y_t &= c_t + i_t \\
c_t, k_{t+1} &\geq 0, k_0 \; given
\end{aligned}
$$

The Deterministic Growth Model in Two Forms

Sequential form and Bellman (recursive) form.
With full depreciation assumption $\delta = 1$:

$$
\begin{aligned}
& V^*(k_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
s.t. \qquad & k_{t+1} \in \Gamma(k_t) \\
& k_0 \in X given
\end{aligned}
$$

Bellman's Principle of Optimality implies we can write this:

$$
V(k) = \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta V(k')\}
$$

$$
for\ all\ k \in X, k_0\ given.
$$

Envelop Theorem in our simplest setting

We assume that $f(k) = k^\alpha$, and depreciation rate $\delta = 1$
We want to choose k' to maximize

$$U = u(k^\alpha - k') + \beta V(k')$$

First order condition:

$$0 = -u'(k^\alpha - k') + \beta dV(k')/dk'$$

We write the value function as:

$$V(k) = \max_{k'}\{u(k^\alpha - k') + \beta V(k')\}$$

Envelop Theorem in our simplest setting

$$V(k) = u(k^\alpha - k') + \beta V(k')$$

$$\begin{aligned}
dV(k)/dk &= \alpha k^{\alpha-1} u'(k^\alpha - k') - u'(k^\alpha - k')\frac{dk'}{dk} + \beta \frac{dV(k')}{dk'}\frac{dk'}{dk} \\
&= \alpha k^{\alpha-1} u'(k^\alpha - k') + \{-u'(k^\alpha - k')\frac{dk'}{dk} + \beta \frac{dV(k')}{dk'}\}\frac{dk'}{dk} \\
&= \alpha k^{\alpha-1} u'(c)
\end{aligned}$$

Stochastic Dynamic Programming

Our goal: to set-up and solve a problem like this

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \{F(k,k',z) + \beta E[V(k',z')|z]\}$$

z_t is a stochastic component.
We need to specify some stochastic process for z_t.

最优化的两个例子：离散时间和连续时间

离散时间的例子

$$
\begin{aligned}
& \max_{k_t,c_t} \sum_{t=0}^{\infty} \beta^t log c_t \\
s.t. \qquad & k_{t+1} = k_t^{\alpha} - c_t \\
& k_0 > 0
\end{aligned}
$$

重新写这个问题：

$$V(x) = \max_{y \geq 0} log(x^{\alpha} - y) + \beta V(y)$$

x是今天的资本存量，y是明天的资本存量。

离散时间的例子

一阶条件:

$$\frac{1}{x^{\alpha} - y} = \beta V'(y)$$

由包络定理可得:

$$V'(x) = \frac{\alpha x^{\alpha-1}}{x^{\alpha} - y}$$

以上两式,并假设$y = \kappa x^{\alpha}$

$$\frac{1}{x^{\alpha} - \kappa x^{\alpha}} = \frac{\alpha(\kappa x^{\alpha})^{\alpha-1}}{(\kappa x^{\alpha})^{\alpha} - \kappa(\kappa x^{\alpha})^{\alpha}}$$

离散时间的例子

以上可得：

$$\kappa = \alpha\beta$$

因此：

$$\begin{aligned} k_{t+1} &= \alpha\beta k_t^{\alpha} \\ c_t &= (1-\alpha\beta)k_t^{\alpha} \end{aligned}$$

连续时间的例子

Pretty much all deterministic optimal control problem in continuous time can be written as

$$V(x_0) = \max_{\{y_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} f(x_t, y_t) dt$$

subject to the law of motion for the state

$$\dot{x}_t = g(x_t, y_t) \; and \; y_t \in A.$$

for $t \geq 0$ and x_0 given.

- $\rho \geq 0$: discount factor;
- $x \in X$: state vector, it can be multi-dimensional, we only need to know the one dimensional case;
- $y \in A$: control vector;
- $f : X \times A \to \mathbb{R}$: instantaneous return function.

连续时间的例子

- Can obtain necessary and sufficient conditions for an optimum using the following procedure(cookbook):
- Current-Value Hamiltonian:

$$H(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

  λ: co-state vector having the same dimension as the state vector
- Necessary and sufficient conditions:

$$\begin{aligned} H_y(x_t, y_t, \lambda_t) &= 0; \\ \dot{\lambda}_t &= \rho\lambda_t - H_x(x_t, y_t, \lambda_t); \\ \dot{x}_t &= g(x_t, y_t) \end{aligned}$$

  for all l$t \geq 0$.
- Initial state x_0 is given.
- Boundary condition for co-state variable λ_t, called "transversality condition"

$$\lim_{T\to\infty} e^{-\rho T}\lambda_T x_T = 0$$