Lecture 6 Basics for Dynamic Stochastic General-Equilibrium Models

21-31

DSGE

Dynamic stochastic general equilibrium

Article Talk

From Wikipedia, the free encyclopedia

Dynamic stochastic general equilibrium modeling (abbreviated as **DSGE**, or **DGE**, or sometimes **SDGE**) is a macroeconomic method which is often employed by monetary and fiscal authorities for policy analysis, explaining historical time-series data, as well as future forecasting purposes.[1] DSGE econometric modelling applies general equilibrium theory and microeconomic principles in a tractable manner to postulate economic phenomena, such as economic growth and business cycles, as well as policy effects and market shocks.

DSGE

- Dynamic: The effect of current choices on future uncertainty makes the models dynamic and assigns a certain relevance to the expectations of agents in forming macroeconomic outcomes.
- Stochastic: The models take into consideration the 传导 transmission of random shocks into the economy and the subsequent economic fluctuations.
- General: referring to the entire economy as a whole (within the model) in that price levels and output levels are determined jointly. As opposed to a Partial equilibrium where price-levels are taken as given and only output-levels are determined within the model economy.
- Equilibrium: Subscribing to the Walrasian, General Competitive Equilibrium Theory, the model captures the interaction between policy actions and subsequent behaviour of agents'

DSGE

Real Business Cycle

Schools [edit]

Two schools of analysis form the bulk of DSGE modeling:[note 4] the classic RBC models, and the New-Keynesian DSGE models that build on a structure similar to RBC models, but instead assume that prices are set by monopolistically competitive firms, and cannot be instantaneously and costlessly adjusted. Rotemberg & Woodford introduced this framework in 1997. Introductory and advanced textbook presentations of

Outline

Last time, we discussed Real Business Cycle and New Keynesian models. We didn't emphasize how solve the model in DSGE framwork.

- Set up and solve the problem with DSGE (the stochastic growth model)

Reading:
Ljungqvist and Sargent Chapter 2, 12
Adda and Cooper Chapter 5
Kydland and Prescott 1982

Plan for today

- Deterministic Growth
- Stochastic Process
- How to solve the stochastic model

The Deterministic Growth Model

确定性的

$\sum_{t=0}^{\infty} \beta^t u(c_t)$

Our interest is in the problems of the form

$c_t = f(k_t) - I_t$
$= f(k_t) - (k_{t+1} - (1-\delta)k_t)$

$$
\begin{aligned}
V^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = F(k_t, k_{t+1}) \\
s.t. \qquad & x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \cdots \\
& x_0 \in X \ given
\end{aligned}
$$

k_t ↓ 状态变量 k_{t+1} ↓ 选消费多少

控制 c_t ⟹ 决定了下一期状态.

选 $\{c_t, I_t\}$ ⟺ 选 $\{k_{t+1}\}$.

- The mapping Γ: $X \to Y$ is a correspondence: for any $x \in X$ it assigns a set $\Gamma(x) \subset Y$

The Deterministic Growth Model

$$
\begin{aligned}
V^*(k_0) &= \max_{k_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. \qquad k_{t+1} &= (1-\delta)k_t + i_t \\
y_t &= f(k_t) \\
y_t &= c_t + i_t \\
c_t, k_{t+1} &\geq 0, k_0 \; given
\end{aligned}
$$

The Deterministic Growth Model in Two Forms

序列形式

Sequential form and Bellman (recursive) form.
With full depreciation assumption $\delta = 1$:

$$V^*(k_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$s.t. \qquad k_{t+1} \in \Gamma(k_t)$$

$$k_0 \in X given$$

贝尔曼形式

Bellman's Principle of Optimality implies we can write this:

$$V(k) = \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta V(k')\}$$

$$for\ all\ k \in X, k_0\ given.$$

$k' \triangleq k_{t+1}$

Stochastic Dynamic Programming

Our goal: to set-up and solve a problem like this

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \{F(k,k',z) + \beta E[V(k',z')|z]\}$$

z_t is a stochastic component.
We need to specify some stochastic process for z_t.

Markov Chains

- Let's consider cases where the stochastic component can take finitely many values (discrete-state process)

有限状态

- x_t will be a Markov chain:

$$x \in S = \{x_1, x_2, x_3, \cdots x_n\}$$

x_i refers to the realization of an event.

- This means it has the Markov property:

当期状态只取决于上一期

$$Pr(x_{t+1} \in S | x_t, x_{t-1}, \cdots x_{t-k}) = Pr(x_{t+1} | x_t)$$

忽略了历史路径

- Future values depend only on the current value. Useful for using recursive techniques.
- We'll consider time-invariant chains: fixed probabilities of moving from one state to another.

状态转换概率不随时间改变

Markov Chains

- The stochastic process x_t will be a sequence of random vectors.
- The n dimensional state space consists of vector e_i, $i = 1, \cdots, n$.
- $n \times 1$ unit vector that records the position of the system. E.g.

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Markov Chains

$n \times n$ 转换矩阵

- $n \times n$ transition/stochastic matrix P: records the probabilities of moving from one state to another in one period.

$$P_{ij} = Pr(\underbrace{x_{t+1} = e_j}_{\text{明天}j} | \underbrace{x_t = e_i}_{\text{今天}i})$$

从$i \to j$

- To be probabilites, for $i = 1, \cdots, n$, the matrix must satisfy:

$$\sum_{j=1}^{n} P_{ij} = 1$$

$$P_{ij} \geq 0$$

- There needs to be an initial probability distribution: π_0

An Example

- How does Markov chain x_t relate to the state variable we care about, e.g. TFP z_t ? How would you forecast it?
- Suppose GDP growth, y_t, can be in boom or bust.
- The boom state implies $y_t = 1.2$ and the recession state $y_t = -0.4$
- x_1 indicates we are in a boom, x_2, in a recession.
- e.g Hamilton (1989):

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$$

Probability distribution over time

$(\alpha, \beta)\begin{pmatrix} p_{11}, & p_{12} \\ p_{21}, & p_{22} \end{pmatrix} = (\alpha p_{11} + \beta p_{21}, \ \alpha p_{12} + \beta p_{22})$

- Define π_t^T as the unconditional probability distribution of x_t,$(1 \times n)$ whose i^{th} element is $Pr(x_t = e_i)$.
- From an initial distribution $\pi_0^T = (0, \cdots, 1, \cdots, 0)$, the probability distribution of x_1 is:

转置

$$\pi_1^T = Pr(x_1) = \pi_0^T P$$

and for x_2 it's:

$$\pi_2^T = Pr(x_2) = \pi_1^T P = (\pi_0^T P)P = \pi_0^T P^2$$

- In general:

$$\pi_k^T = \pi_0^T P^k$$

Stationary Distributions

固定的

- Is there a stationary (or invariant) distribution, π?

$$\pi' = \pi' P$$

  If we start with a distribution over states π, tomorrow we end up with the same distribution over states.

- There's always at least one stationary distribution. It is an eigenvector associated with the unit eigenvalue of P'.

特征向量

$$\pi'(I - P) = 0$$
$$(I - P')\pi = 0$$

- Markov chain (π, P) is stationary if, for a given initial distribution: $\pi' = \pi' P$

Asymptotic Stationarity

- Given a π_0, does π_t approach a stationary distribution over time?
- I.e. $\lim_{t\to\infty} \pi_0 P^t = \pi_\infty$ where π_∞ solves

$$(I - P')\pi = 0$$

- If, for all π_0 $\lim_{t\to\infty} \pi_0 P^t = \pi_\infty$, we say the Markov chain is asymptotially stationary with a unique invariant distribution.
- Will be true, from every state there is a positive probability of moving to another state in one or more steps.

Markov processes vs. Markov chains

- The stochastic process could have a continuous state space.
- We'll see some like this before:

$$z_{t+1} = \rho z_t + \epsilon_t$$

$\mathbb{E}[z_{t+1} | z_t] = \rho z_t$

独立同分布

- If ϵ is i.i.d. then z_t follows a Markov process.
- The conditional expectation depends only on the last realization of the process.
- Computationally it is useful to discretize continuous state Markov processes as a Markov chain.

Approximation of a continuous state Markov process

- Choose some extreme values for the process, e.g. r standard deviations from the mean to set the bounds.
- Discretize the state space into $z = [z_1, \cdots, z_n]$. The distance between each is δ. For any two grid points:

$$\begin{aligned} P_{jk} &= Pr(z_k - \delta/2 < \rho z_j + \epsilon_t < z_k + \delta/2) \\ &= Pr(z_k - \delta/2 - \rho z_j < \epsilon_t < z_k - \rho z_j + \delta/2) \end{aligned}$$

$$P_{jk} = F(\frac{z_k - \rho z_j + \delta/2}{\sigma}) - F(\frac{z_k - \rho z_j - \delta/2}{\sigma})$$

Simple example with i.i.d. shocks

$$Pr(z_t = z^h | z_{t-1} = z^h) = Pr(z_t = z^h | z_{t-1} = z^l) = 0.5$$
$$Pr(z_t = z^l | z_{t-1} = z^h) = Pr(z_t = z^l | z_{t-1} = z^l) = 0.5$$
$$z^h > z^l$$

If we expand the expectation, what does the Bellman equation look like?

代入期望公式

$$V(k, z) = \max_{k' \in \Gamma(k, z_i)} \{F(k, k', z) + \beta \sum_{j=1}^{n} P_{ij} V(k', z'))\}$$
$$V(k, z^h) = \max\{u^h + \beta[P_{hh} V(k', z^h) + P_{hl} V(k', z^l)]\}$$
$$V(k, z^l) = \max\{u^l + \beta[P_{lh} V(k', z^h) + P_{ll} V(k', z^l)]\}$$

Stochastic Dynamic Programming

$$V(k, z_i) = \max_{k' \in \Gamma(k, z_i)} \{F(k, k', z_i) + \beta \sum_{j=1}^{n} P_{ij} V(k', z_j)\}$$

In general, all the proofs you saw for the deterministic case can be applied to the stochastic case.
More generally,

$$V(k, z) = \max_{k' \in \Gamma(k, z)} \{F(k, k', z) + \beta E(V(k', z')|z)\}$$

z could be, e.g., a finite Markov chain or an AR(1) process. (The latter continuous case requires added steps to the proofs, but think of a discrete approximation.)

An Example

$$V(k,z) = \max_{k'}\{u(zk^{\alpha} - k' + (1-\delta)k) + \beta E\{V(k',z')|z\}\}$$

for all (z,k).

- z is a bounded, random variable that follows a first order Markov process (e.g. an AR(1) process).
- There exists a maximum possible capital stock such that consumption is non-negative. Provided there is discounting $\beta < 1$ and that the shocks follows a bounded first-order Markov process, there exists a unique value function.
- We are interested in finding the policy function: $k' = \phi^k(k,z)$

Is there a steady state

- If z_t does not have a degenerate distribution, k_t will not converge to a single number: $k' = \phi_k(z, k)$, why?
- It will converge to an invariant limiting distribution. 收敛到不变的极限分布
- At sufficiently far away horizons, k should be independent of k_0. k与k0独立. T充分大
- The average value in this distribution will be the same as the time average of k_t as $T \to \infty$. 均值不变，T→∞均值仍不变.
- The stochastic process for the capital stock is therefore ergodic. 各态历经的
- Instead of a steady state, we have an invariant limiting distribution for capital, output etc. 不变的

Solving the stochastic growth model

Consider the stochastic growth model again:

$$V(k,z) = \max_{k'}\{u(zk^{\alpha} - k' + (1-\delta)k) + \beta E V(k', z')|z\}$$

for all (z,k).
We want to solve the model to find:
The value function itself.
The policy functions:

$$c = \phi^{c}(k,z)$$

$$k' = \phi^{k}(k,z)$$

we want to solve for the endogenous variables only as functions of the state each period (and the deep parameters).

Solution Methods

- Guess and Verify: only works in limited cases

不考
- Value function iteration

  线性化　　待定系数法
- Linearizion: undetermined coefficients and eigenvalue decomposition. 特征值分解

Guess and Verify

- If we know what form the solution takes we can use this as a guess, find the unknown coefficients and verify it is a solution.
- Two options:
  - Guess and verify the value function, deriving the policy function along the way.
  - Guess and verify the policy function directly.
- Works well, but only for very special cases $u(c) = lnc$ and $\delta = 1$.

Detour: Envelop Theorem in our simplest setting

We assume that $f(k) = k^\alpha$, and depreciation rate $\delta = 1$
We want to choose k' to maximize

$$U = u(k^\alpha - k') + \beta V(k')$$

First order condition: 对k'求导 ⇒ 选择k'

$$0 = -u'(k^\alpha - k') + \beta dV(k')/dk'$$

We write the value function as:

$$V(k) = \max_{k'} \{u(k^\alpha - k') + \beta V(k')\}$$

Detour: Envelop Theorem in our simplest setting

包络定理

$$V(k) = u(k^\alpha - k') + \beta V(k')$$

$$dV(k)/dk = \alpha k^{\alpha-1} u'(k^\alpha - k') - u'(k^\alpha - k')\frac{dk'}{dk} + \beta \frac{dV(k')}{dk'}\frac{dk'}{dk}$$

$= \alpha k^{\alpha-1} u'(k^\alpha - k') + \{-u'(k^\alpha - k')$ [redacted] $+ \beta \frac{dV(k')}{dk'}\}\frac{dk'}{dk}$

$$= \alpha k^{\alpha-1} u'(c)$$

代入 FOC. ↗

$$0 = -u'(k^\alpha - k') + \beta \frac{dV(k')}{dk'}$$

Guess and verify the policy function

- Assume log utility and $\delta = 1$.
- Let's make an (informed!) guess that the policy function for k' takes the form:

$$k' = Qzk^{\alpha}$$

- We'll also make use of the stochastic Euler derived [Envelope Theorem Used]:

$k' = Qzk^{\alpha}$

$$\frac{1}{c} = \beta E\left(\frac{\alpha z'(k')^{\alpha-1}}{c'} | z\right)$$

$u(zk^{\alpha} - k') + \beta E(V(k', z')|z)$

and the resource constraint:

$\{k'\}: -\frac{1}{c} + \beta E \frac{dV(k', z')|z)}{dk'}$

$$c = zk^{\alpha} - k'$$

$V(k) = \max_{k'}\{\cdots\}$

- From the resource constraint, the policy function for consumption is

$V(k') = \max_{k''}\{\cdots\}$

$$c = (1 - Q)zk^{\alpha}$$

$-\frac{1}{c} + \beta E\left(\frac{\alpha z'(k')^{\alpha-1}}{c'}\right)$

Guess and verify

$$\frac{1}{(1-Q)zk^{\alpha}} = \beta E\{\frac{\alpha z'(k')^{\alpha-1}}{(1-Q)z'k'^{\alpha}}|z\}$$

$$\frac{1}{zk^{\alpha}} = \beta\alpha(k')^{-1}$$

$$k' = \alpha\beta z k^{\alpha}$$

$k' = \alpha\beta z k^{\alpha}$

$Q = \alpha\beta$

α: Capital Share

β = Discount Rate

Guess and verify the value function

Guess a form for the value function

$$V(k,z) = G + Bln(k) + Dln(z)$$

- Use the first order conditions from the Bellman equation and from the guess with respect to k' to find the general form of the policy function.
- Substitute this, and our guess into the Bellman equation.
- Solve for the unknowns G, B, D
- Details

stop here //

Value function iteration

- The value function is a vector of optimal utility values for each (k, z).
- Make an initial guess of the value function, V_0, can even be $V_0 = 0, \forall(k, z)$.
- Plug this into the Bellman equation and find $V_1, \forall(k, z)$:

$$V_1(k, z) = \max F(k, k', z) + \beta E(V_0(k', z')|z)$$

- Check if $|V_1 - V_0| < \epsilon$.
- The value function gives the maximum utility level for all (k, z) pairs. This function should be the same on the left and right hand side.
- If not, iterate:

$$V_2(k, z) = \max F(k, k', z) + \beta E(V_1(k', z')|z)$$

Value function iteration: an example with Matlab code

This part is from Eric Sims notes. Value function iteration. For showing you the coding process, we start again from the deterministic model.

- $$V(k) = \max_c \{ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k') \}$$
  $$s.t \;\; k' = k^\alpha - c + (1-\delta)k$$
- Solve the steady state k.

Value function iteration: an example with Matlab code

Now the stochastic model.

- $$V(k,A) = \max_c \{\frac{c^{1-\sigma}}{1-\sigma} + \beta V(k',A')\}$$
  $$k' = Ak^\alpha - c + (1-\delta)k$$

- $$A = \begin{bmatrix} 0.9 \\ 1.0 \\ 1.1 \end{bmatrix}$$

- $$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

VFI CODING

PLEASE REFER TO CODE.PDF, WE GO OVER THE DETAILS AS WE CHECK THE CODE.

Linearization: an example

Consider all the equilibrium conditions together:

$$
\begin{aligned}
y_t &= z_t k_t^{\alpha} \\
y_t &= c_t + i_t \\
k_{t+1} &= (1-\delta)k_t + i_t \\
\frac{1}{c_t} &= \beta E_t(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)\frac{1}{c_{t+1}}
\end{aligned}
$$

We could linearize all these equations around the deterministic steady state and consider perturbations.

The linearized system

$$\begin{aligned}
\hat{y_t} &= \hat{z_t} + \alpha \hat{k_t} \\
y\hat{y_t} &= c\hat{c_t} + i\hat{i_t} \\
k\hat{k_{t+1}} &= (1-\delta)k\hat{k_t} + i\hat{i_t} \\
E_t\hat{c_{t+1}} &= \hat{c_t} + E_t\hat{r_{t+1}}
\end{aligned}$$

where

$$\begin{aligned}
\hat{r_t} &= \frac{r-1+\delta}{r}(\hat{z} - (1-\alpha)\hat{k}) \\
\hat{z} &= \rho\hat{z_{t-1}} + \epsilon_t
\end{aligned}$$

Variables without a hat are steady state values. Some of these can be found, some have to be "calibrated".

The steady state

We'll fix values for $\beta, \alpha, \delta, z, \rho$.
From the Euler equation:

$$r = \frac{1}{\beta}$$

Which implies, from the definition of the return on capital:

$$\frac{1}{\beta} = \alpha z k^{\alpha-1} + 1 - \delta$$

So we can solve for the steady state capital stock. Steady state output is then simply:

$$y = zk^{\alpha}$$
$$i = \delta k$$
$$c = y - i$$

Solution Method

The collection of linearized conditions can be written recursively:

$$AE_t \begin{bmatrix} \hat{x_{t+1}} \\ \hat{w_{t+1}} \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix}$$

where x is the vector of state variables. w is the vector of control variables.
The solution to this linear rational expectations model is then of the form:

$$\hat{w}_t = F\hat{x}_t$$
$$\hat{x_{t+1}} = P\hat{x}_t$$

These two equations are the linear policy functions: e.g. they are of the same form as before $c = \phi_c(k, z)$ and $k' = \phi_k(k, z)$

Blanchard-Kahn and eigenvalue decomposition

$$E_t A \begin{bmatrix} \hat{x_{t+1}} \\ \hat{w_{t+1}} \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix}$$

Blanchard-Kahn conditions for $B^{-1}A$:

- Unstable eigenvalues$=$ number of controls (jumps), and stable $=$ number of states $\rightarrow$ unique solution.
- Too many unstable: explosive solution
- Too few unstable: multiple equilibria

Blanchard and Kahn (1980)

- Decompose $C = B^{-1}A = P^{-1}\Lambda P$: Λ contains the eigenvalues, P the eigenvectors.
- Partition Λ to match x, w:

$$\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} E_t \begin{bmatrix} \hat{x_{t+1}} \\ \hat{w_{t+1}} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix}$$

- Define new variables:

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{w}_t \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix}$$

then

$$\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} E_t \begin{bmatrix} \tilde{x_{t+1}} \\ \tilde{w_{t+1}} \end{bmatrix} = \begin{bmatrix} \tilde{x}_t \\ \tilde{w}_t \end{bmatrix}$$

Blanchard and Kahn (1980)

- For the transformed controls, Λ_2 contains unstable eigenvalues. The only solution satisfying TVC is $\tilde{w_{t+s}} = 0$. This implies:

$$P_{21}\hat{x}_t + P_{22}\hat{w}_t = 0$$
$$\hat{w}_t = -P_{22}^{-1}P_{21}\hat{x}_t$$

- Substitute this expression into the definition of $\tilde{x}_t$:

$$\tilde{x}_t = P_{11}\hat{x}_t + P_{12}\hat{w}_t = (P_{11} - P_{12}P_{22}^{-1}P_{21})\hat{x}_t = Q\hat{x}_t$$

and therefore

$$E_t\hat{x_{t+1}} = Q^{-1}\Lambda_1^{-1}Q\hat{x}_t$$

Impulse response functions

$$c_t = F \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$
$$\begin{bmatrix} k_{t+1} \\ z_{t+1} \end{bmatrix} = P \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

We can "shock" the system in period 1 by setting $z_t = 1$.
How does the system evolve afterwards? This is called an impulse response function.

Undetermined Coefficients

We are looking for a solution of the form:

$$\hat{c}_t = F_k \hat{k}_t + F_z \tilde{z}_t$$
$$\hat{k_{t+1}} = P_{kk} \hat{k}_t + P_{kz} \tilde{z}_t$$

Our linearized system can be written as:

$$\begin{aligned}
\hat{k_{t+1}} &= \lambda_1 \hat{k}_t + \lambda_2 \tilde{z}_t + (1 - \lambda_1 - \lambda_2)\hat{c}_t \\
E_t c_{t+1} &= \hat{c}_t + E_t \hat{r_{t+1}} \\
\hat{r}_t &= \lambda_3 (\tilde{z}_t - \hat{k}_t) \\
\tilde{z}_t &= \rho \tilde{z_{t-1}} + \epsilon_t
\end{aligned}$$

Undetermined Coefficients

$$
\begin{aligned}
\hat{k_{t+1}} &= \lambda_1 \hat{k_t} + \lambda_2 \tilde{z}_t + (1-\lambda_1-\lambda_2)\hat{c}_t \\
E_t c_{t+1} &= \hat{c}_t + E_t \hat{r_{t+1}} \\
\hat{r}_t &= \lambda_3(\tilde{z}_t - \hat{k_t})
\end{aligned}
$$

Substitute the guess for the consumption function into the capital accumulation equation and equate coefficients:

$$
\begin{aligned}
P_{kk} &= \lambda_1 + (1-\lambda_1-\lambda_2)F_k \\
P_{kz} &= \lambda_2 + (1-\lambda_1-\lambda_2)F_z
\end{aligned}
$$

Undetermined Coefficients

Now, substitute the guesses into the Euler equation:

$$-(F_k\hat{k_t} + F_z\tilde{z}_t) = E_t\hat{r_{t+1}} - E_t(F_k\hat{k_{t+1}} + F_z\tilde{z_{t+1}})$$

Substitute the definition of MPK, the TFP process

$$\begin{aligned} -(F_k\hat{k_t} + F_z\tilde{z}_t) &= E_t(\lambda_3\rho\tilde{z}_t - \lambda_3 P_{kk}\hat{k_t} - \lambda_3 P_{kz}\tilde{z}_t) \\ &\quad - (F_k P_{kk}\hat{k_t} + F_k P_{kz}\tilde{z}_t + F_z\rho\tilde{z}_t) \end{aligned}$$

Equating coefficients

$$\begin{aligned} -F_k &= -\lambda_3 P_{kk} - F_k P_{kk} \\ -F_z &= -\lambda_3 P_{kz} - F_z P_{kz} + \lambda_3\rho - F_z\rho \end{aligned}$$

Undetermined Coefficients

Using the expressions for P_{kk} and P_{kz} in the two equations we just found gives a quadratic expression:

$$Q_1 F_k^2 + Q_2 F_k + Q_3 = 0$$

where

$$\begin{aligned} Q_1 &= 1 - \lambda_1 - \lambda_2 \\ Q_2 &= \lambda_1 - 1 + \lambda_3(1 - \lambda_1 - \lambda_2) \\ Q_3 &= \lambda_3 \lambda_1 \end{aligned}$$

Two solutions: but inspection of capital equation shows that the positive solution is necessary for stability. Can use the solution for F_k to find the other F and P terms.

Remarks

- We've now seen various ways to solve the model.
- VFI is powerful, although harder for very complex models.
- Guess and verify only works in limited cases.
- Linearization is fast and works with a range of more complex models. But is it accurate?

APPENDIX

Appendix for Undetermined Coefficients

The linearized equation system:

$$\begin{aligned}
\hat{y_t} &= \hat{z_t} + \alpha \hat{k_t} \\
y\hat{y_t} &= c\hat{c_t} + i\hat{i_t} \\
k\hat{k_{t+1}} &= (1-\delta)k\hat{k_t} + i\hat{i_t} \\
E_t\hat{c_{t+1}} &= \hat{c_t} + E_t\hat{r_{t+1}}
\end{aligned}$$

starting from the third equation

$$\begin{aligned}
k\hat{k_{t+1}} &= (1-\delta)k\hat{k_t} + i\hat{i_t} \\
\hat{k_{t+1}} &= (1-\delta)\hat{k_t} + \frac{i}{k}\hat{i_t} \\
\hat{k_{t+1}} &= (1-\delta)\hat{k_t} + \delta\hat{i_t}
\end{aligned}$$

The last equation comes from the fact that in steady state $i = \delta k$

Appendix for Undetermined Coefficients

$$y\hat{y}_t = c\hat{c}_t + i\hat{i_t}$$
$$\hat{i_t} = \frac{y}{i}\hat{y}_t - \frac{c}{i}\hat{c}_t$$

combining with

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k_t}$$

we have:

$$\hat{i_t} = \frac{y}{i}(\hat{z}_t + \alpha\hat{k_t}) - \frac{c}{i}\hat{c}_t$$

plug back into

$$\widehat{k_{t+1}} = (1-\delta)\hat{k_t} + \delta\hat{i_t}$$

get

$$\widehat{k_{t+1}} = (1-\delta)\hat{k_t} + \delta[\frac{y}{i}(\hat{z}_t + \alpha\hat{k_t}) - \frac{c}{i}\hat{c}_t]$$

Appendix for Undetermined Coefficients

$$
\begin{aligned}
\hat{k_{t+1}} &= (1-\delta)\hat{k_t} + \delta[\frac{y}{i}(\hat{z}_t + \alpha\hat{k_t}) - \frac{c}{i}\hat{c}_t] \\
&= (1-\delta+\alpha\delta\frac{y}{i})\hat{k_t} + \delta\frac{y}{i}\hat{z}_t - \delta\frac{y-i}{i}\hat{c}_t \\
&= (1-\delta+\alpha\delta\frac{y}{i})\hat{k_t} + \delta\frac{y}{i}(1-\alpha)\frac{\hat{z}_t}{1-\alpha} + \delta(1-\frac{y}{i})\hat{c}_t \\
&= (1-\delta+\alpha\delta\frac{y}{i})\hat{k_t} + \delta\frac{y}{i}(1-\alpha)\tilde{z}_t + \delta(1-\frac{y}{i})\hat{c}_t
\end{aligned}
$$

of which the last equation comes from defining $\tilde{z}_t = \frac{\hat{z}_t}{1-\alpha}$.
now set

$$\lambda_1 = 1 - \delta + \alpha\delta\frac{y}{i}$$

$$\lambda_2 = \delta\frac{y}{i}(1-\alpha)$$

we can show that

$$1 - \lambda_1 - \lambda_2 = 1 - (1-\delta+\alpha\delta\frac{y}{i}) - \delta\frac{y}{i}(1-\alpha) = \delta(1-\frac{y}{i})$$

Appendix for Undetermined Coefficients

re-write

$$\hat{k_{t+1}} = (1 - \delta + \alpha\delta\frac{y}{i})\hat{k_t} + \delta\frac{y}{i}(1-\alpha)\tilde{z}_t + \delta(1 - \frac{y}{i})\hat{c}_t$$
$$= \lambda_1\hat{k_t} + \lambda_1\tilde{z}_t + (1 - \lambda_1 - \lambda_2)\hat{c}_t$$

with

$$\lambda_1 = 1 - \delta + \alpha\delta\frac{y}{i}$$
$$\lambda_2 = \delta\frac{y}{i}(1-\alpha)$$

now we do similar things to another equation

$$\begin{aligned}\hat{r_t} &= \frac{r-1+\delta}{r}(\hat{z}-(1-\alpha)\hat{k}) \\ &= \frac{r-1+\delta}{r}(1-\alpha)(\frac{\hat{z}}{1-\alpha}-\hat{k}) \\ &= \frac{r-1+\delta}{r}(1-\alpha)(\tilde{z}_t-\hat{k}) \\ &= \lambda_3(\tilde{z}_t-\hat{k})\end{aligned}$$

of which $\tilde{z}_t = \frac{\hat{z}_t}{1-\alpha}$ and

$$\lambda_3 = \frac{r-1+\delta}{r}(1-\alpha)$$

and we will cheat by replacing the $\tilde{z}_t$ with $\hat{z}_t$ (This is really bad way to write things, but you get it that what really matters is how we set-up the undetermined coefficient method.) Now we go back to slides page 45.