Assignment 1*

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*Due date: Nov 21st. Please scan your HW solution as PDF file, name it with "your name+ student ID+ HW1", and send it to Hongya's email:liuhongya@smail.nju.edu.cn.
The assignment will be graded based on a "0", "1", "2", and "3". those who get "2" or above will be given full credit in the final evaluation for the "assignment" part.

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1. Government Spending in the Solow Model

Let us introduce government spending in the basic Solow model. Consider the basic model without technological change and suppose the market clearing for the output goes as:

$$Y_t = C_t + I_t + G_t$$

with G_t denoting government spending at time t. Imagine that government spending is given by $G_t = \sigma Y_t$.

(1). Discuss how the relationship between income and consumption should be changed. Is it reasonable to assume that $C_t = sY_t$?

(2). Suppose that government spending partly comes out of private consumption, so that $C_t = (s - \lambda\sigma)Y_t$, where $\lambda \in [0, 1]$. What is the effect of higher government spending (in the form of higher σ) on the equilibrium of the Solow model?

(3). Now suppose that a fraction ϕ of G_t is invested in the capital stock, so that total investment at time t is given by:

$$I_t = (1 - s - (1 - \lambda)\sigma + \phi\sigma)Y_t.$$

Show that if ϕ is sufficiently, the steady-state level of capital-labor ratio will increase as a result of higher government spending (corresponding to higher σ). Is this reasonable? How would you alternatively introduce public investment in this model?

2. Modified Solow Model

Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^{\beta} K^{\alpha} Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha + \beta < 1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

(1). First suppose that there is no population growth. Find the steady-state capital-labor ration in the steady-state output level.

(2). Now suppose that there is population growth at the rate of n, that is $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to returns to land and the wage rate as $t \to \infty$?

(3). Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

3. The Lucas asset-pricing model

The Lucas asset-pricing model. (Lucas, 1978.) Suppose the only assets in the economy are infinitely lived trees. Output equals the fruit of the trees, which is exogenous and cannot be stored; thus $C_t = Y_t$, where Y_t is the exogenously determined output per person and C_t is consumption per person. Assume that initially each consumer owns the same number of trees. Since all consumers are assumed to be the same, this means that, in equilibrium, the behavior of the price of trees must be such that, each period, the representative consumer does not want to either increase or decrease his or her holdings of trees. Let P_t denote the price of a tree in period t (assume that if the tree is sold, the sale occurs after the existing owner receives that period' s output). Finally, assume that the representative consumer maximizes

$$E \sum_{t=0}^{\infty} \frac{lnc_t}{(1+\rho)^t}$$

(1)Suppose the representative consumer reduces his or her consumption in period t by an infinitesimal amount, uses the resulting saving to increase his or her holdings of trees, and then sells these additional holdings in period $t+1$. Find the condition that C_t and expectations involving Y_{t+1}, P_{t+1}, and C_{t+1} must satisfy for this change not to affect expected utility. Solve this condition for P_t in terms of Y_t and expectations involving Y_{t+1}, P_{t+1}, and C_{t+1}.

(2)Assume that $\lim_{s\to\infty} E\frac{p_{t+s}/y_{t+s}}{(1+\rho)^s} = 0$. Given this assumption, iterate your answer to part (a) forward to solve for P_t. （提示：对于所有的s有$c_{t+s} = y_{t+s}$ ）。

(3) Explain intuitively why an increase in expectations of future dividends does not affect the price of the asset.

Assignment 1

Xi Xiang

2024.11.12

1 Government Spending in the Solow Model

Consider the basic Solow model without technological change, where the market clearing for output is given by:

$$Y_t = C_t + I_t + G_t$$

Here, G_t denotes government spending at time t. Suppose that government spending is given by $G_t = \sigma Y_t$.

1.1 Relationship Between Income and Consumption

Discuss how the relationship between income and consumption should be changed. Is it reasonable to assume that $C_t = sY_t$?

Solve. 将 $G_t = \sigma Y_t$ 代入 $Y_t = C_t + I_t + G_t$, 并整理, 有

$$(1-\sigma)Y_t = C_t + I_t \Rightarrow C_t = (1-\sigma)Y_t - I_t \neq sY_t.$$

故不适合表达为消费与总产出之间具有正比例关系, 是应该为线性关系.

若认为 $I = sY$, 则

$$C_t = (1-\sigma-s)Y_t \triangleq s'Y_t.$$

可以表达为正比例关系. □

1.2 Effect of Higher Government Spending on Equilibrium

Suppose that government spending partly comes out of private consumption, so that $C_t = (s-\lambda\sigma)Y_t$, where $\lambda \in [0,1]$. What is the effect of higher government spending (in the form of higher σ) on the equilibrium of the Solow model?

Solve. 将 $C_t = (s-\lambda\sigma)Y_t$ 代入 $Y_t = C_t + I_t + G_t$, 有

$$(1-\sigma)Y_t = (s-\lambda\sigma)Y_t + I_t \Rightarrow (1-s-(1-\lambda)\sigma)Y_t = I_t.$$

均衡时, 有

$$(1-s-(1-\lambda)\sigma)Y_t = I_t = \delta K_t^*.$$

因为 $\lambda \in [0,1]$, 有

$$\frac{\partial K_t^*}{\partial \sigma} = -\frac{(1-\lambda)Y_t}{\delta} < 0.$$

即政府支出增加（$\sigma \uparrow$）, 均衡时的资本存量下降. $\square$

1.3 Effect of Public Investment on Steady-State Capital-Labor Ratio

Now suppose that a fraction ϕ of G_t is invested in the capital stock, so that total investment at time t is given by:

$$I_t = (1-s-(1-\lambda)\sigma + \phi\sigma)Y_t$$

Show that if ϕ is sufficiently, the steady-state level of capital-labor ratio will increase as a result of higher government spending (corresponding to higher σ). Is this reasonable? How would you alternatively introduce public investment in this model?

Solve. 稳态时, 有

$$I_t^* = (1-s-(1-\lambda)\sigma + \phi\sigma)Y_t^* = \delta K_t^*.$$

改写为人均形式, 由 $y_t \triangleq \dfrac{Y_t}{L_t}, \quad k_t \triangleq \dfrac{K_t}{L_t}$, 有

$$\begin{aligned}
&(1-s-(1-\lambda)\sigma + \phi\sigma)y_t^* = \delta k_t^* \\
&\Rightarrow (1-s+(\lambda+\phi-1)\sigma)y_t^* = \delta k_t^*
\end{aligned}$$

求偏导, 有

$$\frac{\partial k_t^*}{\partial \sigma} = \frac{(\lambda+\phi-1)}{\delta}\frac{\partial y_t^*}{\partial k_t^*}.$$

当 λ 足够大时, $\lambda+\phi-1>0$, 上式大于 0, 从而政府支出上升, 人均资本存量（资本-劳动比）上升.

我的分解：G 可以分为两部分, 分别是政府投资 I_t^G , 政府消费 C_t^G , 即 $G_t = I_t^G + C_t^G$. 故总产出可改写为

$$Y_t = (C_t^P + C_t^G) + (I_t^P + I_t^G).$$

但这样写有些过于简单了, 似乎成为了没有政府的二部门经济. 事实上应该考虑政府消费和投资都具有**乘数效应**. 更复杂的模型可以引入以下概念, 但这里略去：

短期效应：政府消费的乘数效应可能在短期内更为显著, 因为它直接增加了总需求.

长期效应：政府投资的乘数效应可能在长期内更为显著, 因为它不仅增加了总需求, 还提高了生产能力和未来的收入.

仅考虑政府消费乘数 $\dfrac{1}{1-\gamma_1}$ 和政府投资乘数 $\dfrac{1}{1-\gamma_2}$. 我们有

$$Y_t = (1 + \frac{1}{1-\gamma_1})C_t + (1 + \frac{1}{1-\gamma_2})I_t.$$

仿照上例, 我们一定有 $I^G \uparrow, K^* \uparrow, Y^* \uparrow; I^C \uparrow, C^* \uparrow, Y^* \uparrow$. □

2 Consider a Modified Solow Model

Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^\beta K^\alpha Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha + \beta < 1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

2.1 First suppose that there is no population growth

First suppose that there is no population growth. Find the steady-state capital-labor ratio and the steady-state output level.

Solve. 将 $Y = F(K, L, Z) = L^\beta K^\alpha Z^{1-\alpha-\beta}$ 改写为人均形式

$$\frac{Y}{L} = \frac{L^\beta K^\alpha Z^{1-\alpha-\beta}}{L} = \left(\frac{K}{L}\right)^\alpha \left(\frac{Z}{L}\right)^{1-\alpha-\beta} \Rightarrow y = k^\alpha z^{1-\alpha-\beta} = f(k).$$

由基本方程, 以及人口增长率 $n = 0$, 有

$$\dot{k} = sf(k) - (n+\delta)k = sf(k) - \delta k.$$

稳态时, $\dot{k} = 0, k = k^*$, 有

$$sf(k^*) = \delta k^* \Rightarrow s(k^*)^\alpha z^{1-\alpha-\beta} = \delta k^*.$$

整理得

$$k^* = \left(\frac{sz^{1-\alpha-\beta}}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

$$y^* = k^{*\alpha} z^{1-\alpha-\beta} = \left(\frac{sz^{1-\alpha-\beta}}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \cdot z^{1-\alpha-\beta}.$$

□

2.2 Now suppose that there is population growth

Now suppose that there is population growth at the rate of n, that is $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to returns to land and the wage rate as $t \to \infty$?

Solve. 将 $Y = F(K, L, Z) = L^{\beta} K^{\alpha} Z^{1-\alpha-\beta}$ 改写为人均形式

$$\frac{Y}{L} = \frac{L^{\beta} K^{\alpha} Z^{1-\alpha-\beta}}{L} = \left(\frac{K}{L}\right)^{\alpha} \left(\frac{Z}{L}\right)^{1-\alpha-\beta} \Rightarrow y = k^{\alpha} z^{1-\alpha-\beta} = f(k).$$

由基本方程, 有

$$\dot{k} = sf(k) - (n+\delta)k.$$

当 $t \to \infty$, 经济体收敛到稳态.

稳态时, $\dot{k} = 0, k = k^*$, 有

$$sf(k^*) = (\delta + n)k^* \Rightarrow s(k^*)^{\alpha} z^{1-\alpha-\beta} = (\delta + n)k^*.$$

整理得

$$k^* = \left(\frac{sz^{1-\alpha-\beta}}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

土地回报率

$$r_Z = MPZ = \frac{\partial F}{\partial Z} = (1-\alpha-\beta) L^{\beta} K^{\alpha} Z^{-\alpha-\beta}.$$

$t \to \infty$, 经济体收敛到稳态, 代入 K^*, 有

$$r_Z^* = (1-\alpha-\beta) L^{\beta} K^{*\alpha} Z^{-\alpha-\beta}.$$

由于 Z 不变,$K = K^*$, 故 $r_Z^* \to \infty$.

工资率

$$w = MPL = \frac{\partial F}{\partial L} = \beta L^{\beta-1} K^{\alpha} Z^{1-\alpha-\beta}.$$

$t \to \infty$, 经济体收敛到稳态, 代入 K^*, 有

$$w^* = \beta L^{\beta-1} K^{*\alpha} Z^{1-\alpha-\beta}.$$

由于 Z 不变,$K = K^*$, $\beta - 1 < 0$, 故 $w^* \to 0$. □

2.3 Would you expect the population growth rate or the saving rate to change over time in this economy?

Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

Solve. 在索洛模型中, 人口增长率 n 恒定, 而储蓄率 s 是外生给定的, 故两者均保持不变.

但实际中人口增长率 n 可能随着经济发展和生活水平的提高而下降, 因为随着收入增加, 人们可能倾向于选择更小的家庭规模. 储蓄率 s 可能随着经济发展而变化. 在初期, 经济发展可能会导致储蓄率上升（因为人们积累财富）, 但随着经济发展到一定阶段, 储蓄率可能会下降（因为人们开始更多地消费和投资于教育、健康等长期福利）. □

3 The Lucas Asset-Pricing Model

The Lucas asset-pricing model. (Lucas, 1978.) Suppose the only assets in the economy are infinitely lived trees. Output equals the fruit of the trees, which is exogenous and cannot be stored; thus $C_t = Y_t$, where Y_t is the exogenously determined output per person and C_t is consumption per person. Assume that initially each consumer owns the same number of trees. Since all consumers are assumed to be the same, this means that, in equilibrium, the behavior of the price of trees must be such that, each period, the representative consumer does not want to either increase or decrease his or her holdings of trees. Let P_t denote the price of a tree in period t (assume that if the tree is sold, the sale occurs after the existing owner receives that period’s output). Finally, assume that the representative consumer maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \frac{\ln C_t}{(1+\rho)^t}$$

3.1 Condition for Equilibrium

Suppose the representative consumer reduces his or her consumption in period t by an infinitesimal amount, uses the resulting saving to increase his or her holdings of trees, and then sells these additional holdings in period $t+1$. Find the condition that C_t and expectations involving Y_{t+1}, P_{t+1}, and C_{t+1} must satisfy for this change not to affect expected utility. Solve this condition for P_t in terms of Y_t and expectations involving Y_{t+1}, P_{t+1}, and C_{t+1}.

Solve. 代表性消费者的效用最大化问题:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \frac{\ln C_t}{(1+\rho)^t}$$

假设在 t 时刻减少了 ε 消费, 并用这些储蓄购买更多的树.

购买的树数量为 $\dfrac{\varepsilon}{P_t}$.

在 $t+1$ 时刻, 这些额外的树产生的收入为 $P_{t+1} \cdot \dfrac{\varepsilon}{P_t}$, 以及这些树产生的果实收入为 $Y_{t+1} \cdot \dfrac{\varepsilon}{P_t}$.

为了使这一操作不影响预期效用, 必须满足:

$$\frac{\mathrm{d}U}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}_t \left(\frac{\ln C_t - \varepsilon}{(1+\rho)^t} + \frac{\ln C_{t+1} + (Y_{t+1} + P_{t+1})\frac{\varepsilon}{P_t}}{(1+\rho)^{t+1}} \right) = 0.$$

解得

$$-\frac{1+\rho}{C_t - \varepsilon} + \mathbb{E}_t \frac{\frac{1}{P_t}(Y_{t+1} + P_{t+1})}{C_{t+1} + (Y_{t+1} + P_{t+1})\frac{\varepsilon}{P_t}} = 0.$$

让 $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} -\frac{1+\rho}{C_t - \varepsilon} + \mathbb{E}_t \frac{\frac{1}{P_t}(Y_{t+1} + P_{t+1})}{C_{t+1} + (Y_{t+1} + P_{t+1})\frac{\varepsilon}{P_t}} = 0.$$

简化得到

$$-\frac{1+\rho}{C_t} + \mathbb{E}_t \frac{\frac{1}{P_t}(Y_{t+1} + P_{t+1})}{C_{t+1}} = 0.$$

进一步简化

$$P_t = \frac{Y_t}{1+\rho} \mathbb{E}_t \frac{Y_{t+1} + P_{t+1}}{C_{t+1}}.$$

□

3.2 Iterating Forward

Assume that $\lim_{s\to\infty} \mathbb{E}_t \frac{[P_{t+s}/Y_{t+s}]}{(1+\rho)^s} = 0$. Given this assumption, iterate your answer to part (a) forward to solve for P_t.

Solve.

$$P_t = \frac{Y_t}{1+\rho} \mathbb{E}_t \frac{Y_{t+1} + P_{t+1}}{C_{t+1}}.$$

我们将 P_{t+1} 进一步展开：

$$P_{t+1} = \frac{Y_{t+1}}{1+\rho} \mathbb{E}_{t+1} \frac{Y_{t+2} + P_{t+2}}{C_{t+2}}.$$

代入到 P_t 的表达式中：

$$P_t = \frac{Y_t}{1+\rho} \mathbb{E}_t \left[\frac{Y_{t+1}}{C_{t+1}} + \frac{P_{t+1}}{C_{t+1}} \right].$$

继续展开 P_{t+1}：

$$P_{t+1} = \frac{Y_{t+1}}{1+\rho} \mathbb{E}_{t+1} \left[\frac{Y_{t+2}}{C_{t+2}} + \frac{P_{t+2}}{C_{t+2}} \right].$$

代入：

$$P_t = \frac{Y_t}{1+\rho} \mathbb{E}_t \left[\frac{Y_{t+1}}{C_{t+1}} + \frac{1}{C_{t+1}} \cdot \frac{Y_{t+1}}{1+\rho} \mathbb{E}_{t+1} \left[\frac{Y_{t+2}}{C_{t+2}} + \frac{P_{t+2}}{C_{t+2}} \right] \right].$$

根据假设 $\lim_{s\to\infty} \mathbb{E}_t \frac{[P_{t+s}/Y_{t+s}]}{(1+\rho)^s} = 0$，我们知道在无限远处的 P_{t+s} 对 P_t 的影响可以忽略不计.
将 P_t 表示为无穷级数的和：

$$P_t = \sum_{s=0}^{\infty} \frac{Y_t}{(1+\rho)^{s+1}} \mathbb{E}_t \prod_{k=1}^{s} \frac{Y_{t+k}}{C_{t+k}}.$$

由于 $\forall k, Y_{t+k} = C_{t+k}$, $\mathbb{E}_t \prod_{k=1}^{s} \frac{Y_{t+k}}{C_{t+k}} = 1$, 而前面部分相当于等比数列求和，化简为：

$$P_t = \frac{Y_t}{(1+\rho)}.$$

□

3.3 Intuitive Explanation

Explain intuitively why an increase in expectations of future dividends does not affect the price of the asset.

Solve. 直观上, 未来股息预期的增加不会影响当前资产价格的原因在于, 资产价格已经完全反映了所有可获得的信息, 包括未来的股息预期. 根据 Lucas 资产定价模型, 资产价格是未来股息的折现值之和. 因此, 如果市场已经正确地预期了未来的股息, 那么即使股息预期增加, 资产价格也已经提前反映了这一变化.

具体来说, 如果市场参与者对未来股息的预期增加, 他们会在当前提高对资产的需求, 从而推高当前的资产价格. 这一过程会持续到资产价格达到一个新的均衡水平, 该水平已经完全反映了未来股息增加的预期. 因此, 一旦实现了新的均衡, 未来股息预期的增加将不会进一步影响资产价格.

简而言之, 资产价格对未来股息的预期具有前瞻性, 因此未来股息预期的变化已经在当前价格中得到了体现. □