Assignment 2*

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*Due date: Dec 5th. Please scan your HW solution as PDF file, name it with "your name+ student ID+ HW2", and send it to Hongya's email:liuhongya@smail.nju.edu.cn.
The assignment will be graded based on a "0", "1", "2", and "3". those who get "2" or above will be given full credit in the final evaluation for the "assignment" part.

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1. A Neoclassical Economy

Consider a neoclassical economy with a representative household with preferences at time $t = 0$ given by

$$\int_0^\infty exp(-\rho t)\frac{c_t^{1-\theta} - 1}{1-\theta}dt.$$

There is no population growth and labor is supplied inelastically. Assume that the aggregate production function is given by $Y_t = F(A_t K_t, L_t)$, where F satisfies the standard assumptions (constant returns to scale, differentiability, and the Inada conditions).

(1) Define a competitive equilibrium for this economy.

(2) Suppose that $A_t = A_0$ for all t, and characterize the steady-state equilibrium. Explain why the steady-state capital-labor ratio is independent of θ.

(3) Now assume that $A_t = exp(gt)A_0$, and show that a BGP (with constant capital share in national income, and constant and equal rates of growth of output, capital, and consumption) exists only if F takes the Cobb-Douglas form, $Y_t = (A_t K_t)^\alpha L_t^{1-\alpha}$.

(4) Characterize the BGP in the Cobb-Douglas case. Derive the common growth rate of output, capital, and consumption.

2. Asset Price with Dividends

Consider a stock that pays dividends of D_t in period t and whose price in period t is P_t. Assume that consumers are risk-neutral and have a discount rate of r; thus they maximize $E[\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t}]$.

(a) Show that equilibrium requires $P_t = E_t[(D_{t+1} + P_{t+1})/(1+r)]$ (assume that if the stock is sold, this happens after that period's dividends have been paid).

(b) Assume that $\lim_{s\to\infty} E_t[P_{t+s}/(1+r)^s] = 0$ (this is a no-bubbles condition; see the next problem). Iterate the expression in part (a) forward to derive an expression for P_t in terms of expectations of future dividends.

3. Asset Price with Bubbles

Consider the setup of the previous problem without the assumption that $\lim_{s\to\infty} E_t[P_{t+s}/(1+r)^s] = 0$.

(a) **Deterministic bubbles.** Suppose that P_t equals the expression derived in part (b) of Problem 2 plus $(1+r)^t b, b > 0$.

(i) Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied?

(ii) Can b be negative? (Hint: Consider the strategy of never selling the stock.)

(b) **Bursting bubbles.** (Blanchard, 1979.) Suppose that P_t equals the expression derived in part (b) of Problem 2 plus q_t, where q_t equals $(1+r)q_{t-1}/\alpha$ with probability α and equals 0 with probability $1-\alpha$.

(i) Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied?

(ii) If there is a bubble at time t (that is, if $q_t > 0$), what is the probability that the bubble has burst by time t + s (that is, that $q_{t+s} = 0$)? What is the limit of this probability as s approaches infinity?

(c) **Intrinsic bubbles.** (Froot and Obstfeld, 1991.) Suppose that dividends follow a random walk: $D_t = D_{t-1} + e_t$, where e is white noise.

(i) In the absence of bubbles, what is the price of the stock in period t?

(ii) Suppose that P_t equals the expression derived in (i) plus b_t, where $b_t = (1+r)b_{t-1} + ce_t$, $c > 0$. Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied? In what sense do stock prices overreact to changes in dividends?

Assignment 2

Xi Xiang

2024.12.1

1 A Neoclassical Economy

Consider a neoclassical economy with a representative household with preferences at time $t = 0$ given by:

$$\int_0^\infty \exp(-\rho t)\frac{c_t^{1-\theta}-1}{1-\theta}\,dt.$$

There is no population growth and labor is supplied inelastically. Assume that the aggregate production function is given by $Y_t = F(A_t K_t, L_t)$, where F satisfies the standard assumptions (constant returns to scale, differentiability, and the Inada conditions).

1.1 Competitive Equilibrium Definition

Define a competitive equilibrium for this economy.

Solve. 竞争均衡是存在这样一个序列 $\{C_t, k_t, w_t, R_t\}_{t=0}^\infty$, 满足:

1. 代表性消费者在预算约束下最大化效用.
2. 厂商最大化利润.
3. 市场出清.

表达为

$$\begin{aligned}
&\max_{c_t}\int_0^\infty \exp(-\rho t)\frac{c_t^{1-\theta}-1}{1-\theta}\mathrm{d}t\\
s.t.\quad &\dot{a_t} = r_t a_t + w_t - c_t\\
&\lim_{t\to\infty} a_t \exp(\int_0^t r_s \mathrm{d}s) \geq 0.
\end{aligned}$$

□

1.2 Steady-State Equilibrium

Suppose that $A_t = A_0$ for all t, and characterize the steady-state equilibrium. Explain why the steady-state capital-labor ratio is independent of θ.

Solve. 定义 $k_t \triangleq \dfrac{K_t}{L}$.
我们有

$$y_t \triangleq \frac{Y_t}{L} = \frac{F(A_tK_t, L)}{L} = F(\frac{A_tK_t}{L}, 1) \triangleq f(A_tk_t).$$

基本方程

$$\dot{k}_t = f(A_tk_t) - \delta k_t - c_t.$$

租金

$$r_t = MPR = F_k[A_tK_t, L] = A_tf'(A_tk_t) - \delta.$$

Hamitonian 函数

$$H(a, c, \mu) = u(c_t) + \mu_t[r_ta_t + w_t - c_t].$$

FOCs

$$\begin{aligned}
&H_c(a, c, \mu) = u'(c_t) - \mu_t \\
&H_a(a, c, \mu) = \mu_tr_t = -\dot{\mu_t} + \rho\mu_t \\
&\dot{a_t} = r_ta_t + w_t - c_t \\
&\lim_{t\to\infty} a_t \exp(-\rho t)\mu_t = 0
\end{aligned}$$

$$\Rightarrow \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(A_tf'(A_tk_t) - \delta - \rho).$$

其中 $\theta \triangleq -\dfrac{c_tu''(c_t)}{u'(c_t)}$.
稳态时, 有 $\dfrac{\dot{c}_t}{c_t} = 0, \dot{k}_t = 0$.
故

$$\begin{aligned}
&A_tf'(A_tk_t) - \delta - \rho = 0 \\
&f(A_tk_t) - \delta k_t - c_t = 0 \\
&A_t = A_0.
\end{aligned}$$

解得 $A_0f'(A_0k^*) = \delta + \rho$, k^* 与 θ 无关. $c^* = f(A_0k^*) - \delta k^*$, c^* 与 θ 无关.

θ 是替代弹性的倒数, 即它调节了个人在今天消费和未来消费之间的替代意愿. 这个经济没有在稳定状态下的增长. 因此, 消费随着时间的推移是恒定的, 因此一旦达到稳定状态, 消费者对跨期替代的偏好并不重要. □

1.3 Balanced Growth Path (BGP) Conditions

Now assume that $A_t = \exp(gt)A_0$, and show that a BGP (with constant capital share in national income, and constant and equal rates of growth of output, capital, and consumption) exists only if F takes the Cobb-Douglas form, $Y_t = (A_tK_t)^\alpha L_t^{1-\alpha}$.

Solve. 事实上题目要求的是必要条件, 但说明不采用柯布-道格拉斯形式的生产函数就不存在 BGP 比较困难, 所以直接用充要条件蕴含之.

存在 BGP $\Leftrightarrow \dfrac{\dot{c}_t}{c_t}$ 为常数, 且 $\alpha_k = \dfrac{k_t}{f(A_t k_t)}$ 为常数, 且 $g_k = g_y$.
由

$$\begin{aligned} g_c &= \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(A_t f'(A_t k_t) - \delta - \rho)\text{恒定} \Rightarrow A_t f'(A_t k_t) = c_1 \\ \alpha_k &= \frac{k_t}{f(A_t k_t)} = c_2. \end{aligned}$$

其中 $c_1, c_2 \in \mathbb{R}$.
两式相乘, 有

$$\frac{A_t k_t f'(A_t k_t)}{f(A_t k_t)} = c_1 c_2.$$

设 $\chi_t \triangleq A_t k_t, \alpha = c_1 c_2$, 有

$$g_\chi = \frac{\chi_t f'(\chi_t)}{f(\chi_t)} = \alpha.$$

由 $A_t f'(\chi_t) = c_1$, 且 $f'(\cdot)$ 单调递减

$$\chi_t = f'^{-1}\left(\frac{c_1}{A_t}\right).$$

这说明 χ_t 增速应该等于 A_t 的增速.
由 $\chi_t f'(\chi_t) - \alpha f(\chi_t) = 0$, 这是一个关于 χ_t 的微分方程, 解得

$$f(\chi_t) = C\chi_t^\alpha.$$

其中 $C \in \mathbb{R}$.

$$Y_t = L \cdot f(\chi_t) = CL(\chi_t)^\alpha = CL(A_t k_t)^\alpha = C(A_t K_t)^\alpha L^{1-\alpha}.$$

标准化 $C = 1$, 正是柯布-道格拉斯形式的生产函数. □

1.4 Characterization of the BGP in the Cobb-Douglas Case

Characterize the BGP in the Cobb-Douglas case. Derive the common growth rate of output, capital, and consumption.

Solve. 定义 $y_t \triangleq \dfrac{Y_t}{L} = (A_t k_t)^\alpha$.

$$g_y = \frac{\dot{y}_t}{y_t} = \frac{\mathrm{d}\ln(f(\cdot))}{\mathrm{d}t} = \alpha\left(\frac{\dot{A}_t}{A_t} + \frac{\dot{k}_t}{k_t}\right) = \alpha(g + g_k).$$

再由 $g_k = g_y$, 解得

$$g_y = g_k = \frac{\alpha}{1-\alpha}g.$$

由资本积累方程

$$\dot{k}_t = y_t - c_t + (1-\delta)k_t \Rightarrow \frac{c_t}{k_t} = 1 - \delta - g_k + c\text{为常数}.$$

故 $g_c = g_k$.

$$g_k = g_c = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(A_t f'(A_t k_t) - \delta - \rho) = \frac{1}{\theta}(\alpha A_t^{\alpha} k_t^{\alpha-1} - \delta - \rho).$$

由于 A_t, k_t 都在增长, 我们将其标准为 $\chi_t \triangleq \frac{k_t}{A_t^{\frac{\alpha}{1-\alpha}}}$, 有

$$g_k = \frac{1}{\theta}(\alpha \chi_t^{\alpha-1} - \delta\rho) = g_y.$$

BGP 上的 χ_t 满足

$$\chi_t = \chi^* = \left(\frac{\alpha}{\theta g_y + \delta + \rho}\right)^{\frac{1}{1-\alpha}}.$$

k_t, c_t, y_t 以增长速率 g_k 增长的平衡路径, 是变换后变量 χ_t 的稳态, 则

$$k_t = A_t^{\frac{\alpha}{1-\alpha}} \chi^*.$$

验证 BGP

$$\frac{\dot{k}_t}{k_t} = \frac{\alpha}{1-\alpha}\frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{\alpha}{1-\alpha} g = g_k.$$

□

2 Asset Price with Dividends

Consider a stock that pays dividends of D_t in period t and whose price in period t is P_t. Assume that consumers are risk-neutral and have a discount rate of r; thus they maximize $\mathbb{E}\left[\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t}\right]$.

2.1 Equilibrium Condition

Show that equilibrium requires $P_t = \mathbb{E}_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]$.

Solve. 假设 t 期减少一个小的消费 ε, 用于购买 $\frac{\varepsilon}{P_t}$ 数量的股票, t 期效用减少 ε, $t+1$ 期可获得分红 $D_{t+1}\frac{\varepsilon}{P_t}$, 卖出股票收益 $P_{t+1}\frac{\varepsilon}{P_t}$. 均衡时, t 期减少消费带来的负效用等于预期贴现后 $t+1$ 期的收益. 因此

$$\varepsilon = \mathbb{E}_t\left[\left(\frac{1}{1+r}\right)(D_{t+1} + P_{t+1})\left(\frac{\varepsilon}{P_t}\right)\right].$$

整理得

$$P_t = \mathbb{E}_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right].$$

□

2.2 No-Bubbles Condition

Assume that $\lim_{s\to\infty}\mathbb{E}_t\left[\frac{P_{t+s}}{(1+r)^s}\right]=0$ **(this is a no-bubbles condition). Iterate the expression in part (a) forward to derive an expression for** P_t **in terms of expectations of future dividends.**

Solve. 利用上题结论

$$P_t=\mathbb{E}_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]\qquad P_{t+1}=\mathbb{E}_{t+1}\left[\frac{D_{t+2}+P_{t+2}}{1+r}\right]$$

代入, 我们有

$$P_t=\mathbb{E}_t\left[\frac{D_{t+1}}{1+r}\mathbb{E}_{t+1}\left[\frac{D_{t+2}+P_{t+2}}{(1+r)^2}\right]\right]=\mathbb{E}_t\left[\frac{D_{t+1}}{1+r}\right]\mathbb{E}_t\mathbb{E}_{t+1}\left[\frac{D_{t+2}+P_{t+2}}{(1+r)^2}\right].$$

由迭代期望定律[1]

$$P_t=\mathbb{E}_t\left[\frac{D_{t+1}}{1+r}\right]\mathbb{E}_t\left[\frac{D_{t+2}+P_{t+2}}{(1+r)^2}\right].$$

向前迭代到无穷处, 有

$$P_t=\mathbb{E}_t\left[\sum_{s=1}^{\infty}\frac{D_{t+s}}{(1+r)^s}\right]+\lim_{s\to\infty}\mathbb{E}_t\left[\frac{P_{t+s}}{(1+r)^s}\right].$$

再由题设条件 $\lim_{s\to\infty}\mathbb{E}_t\left[\frac{P_{t+s}}{(1+r)^s}\right]=0$, 我们有

$$P_t=\mathbb{E}_t\left[\sum_{s=1}^{\infty}\frac{D_{t+s}}{(1+r)^s}\right].$$

□

3 Asset Price with Bubbles

Consider the setup of the previous problem without the assumption that $\lim_{s\to\infty}\mathbb{E}_t\left[\frac{P_{t+s}}{(1+r)^s}\right]=0$.

3.1 Deterministic Bubbles

Suppose that P_t **equals the expression derived in part (b) of Problem 2 plus** $(1+r)^t b$**,** $b>0$**.**

3.1.1 First-Order Condition

Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied?

[1] 定理: 对于条件期望的运算, 有

$$E(Y)=E_X[E(Y|x)]$$

上式表明, 无条件期望 $E(Y)$ 等于对于给定 $X=x$ 情况下 Y 的条件期望 $E(Y|x)$ 再对 X 求期望.

Solve. 由题设

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + (1+r)^t b.$$

由一阶条件

$$P_t = \mathbb{E}_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right].$$

即证

$$\sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + (1+r)^t b = \mathbb{E}_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right].$$

由题设, $t+1$ 期, 我们有

$$P_{t+1} = \sum_{s=1}^{\infty} \mathbb{E}_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + (1+r)^{t+1} b.$$

两边同时比上 $1+r$, 并取 t 时刻的期望. 利用期望迭代定律, 有

$$\mathbb{E}_t \left[\frac{P_{t+1}}{1+r} \right] = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + (1+r)^t b.$$

故

$$\begin{aligned}
P_t &= \mathbb{E}_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right] = \mathbb{E}_t \left[\frac{D_{t+1}}{1+r} \right] + \mathbb{E}_t \left[\frac{P_{t+1}}{1+r} \right] \\
&= \mathbb{E}_t \left[\frac{D_{t+1}}{1+r} \right] + \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + (1+r)^t b \\
&= \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + (1+r)^t b.
\end{aligned}$$

成立. 故满足原消费者的一阶条件. □

3.1.2 Negative Bubbles

Can b be negative? (Hint: Consider the strategy of never selling the stock.)

Solve. 如果 b 是负的, 那么泡沫项将趋于负无穷, 因此股票的价格最终会变成负的, 并趋于无穷大. 这种情况是不可能的. 股票永远不会以负的价格出售, 持有股票而不出售的策略可以避免以负价格出售的资本损失. 或者更简单地说, 直接对这支股票放任不管, 而不是以负价格出售. 因此 b 不能是负的. □

3.2 Bursting Bubbles

Suppose that P_t equals the expression derived in part (b) of Problem 2 plus q_t, where q_t equals $\frac{(1+r)q_{t-1}}{\alpha}$ with probability α and equals 0 with probability $1-\alpha$.

3.2.1 First-Order Condition

Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied?

Solve. 由题设, 此时资产价格

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + q_t.$$

其中 q_t 的概率分布为

q_t	$(1+r)\frac{q_{t-1}}{\alpha}$	0
P	α	$1-\alpha$

同样地, $t+1$ 期

$$P_{t+1} = \sum_{s=1}^{\infty} \mathbb{E}_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + q_{t+1}.$$

对两边取 t 期时的期望, 并运用期望迭代定律, 我们有

$$\mathbb{E}_t \left[P_{t+1} \right] = \sum_{s=1}^{\infty} \mathbb{E}_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + \frac{(1+r)q_t}{\alpha}\alpha + 0 \cdot (1-\alpha) = \sum_{s=1}^{\infty} \mathbb{E}_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + (1+r)q_t.$$

两边同时比上 $1+r$, 再加上 $\mathbb{E}_t \frac{D_{t+1}}{1+r}$, 有

$$\mathbb{E}_t \left[\frac{D_{t+1}+P_{t+1}}{1+r} \right] = \mathbb{E}_t \left[\frac{D_{t+1}}{1+r} \right] + \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + q_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + q_t.$$

即得证满足原消费者的一阶条件. □

3.2.2 Burst Probability

If there is a bubble at time t (that is, if $q_t > 0$), what is the probability that the bubble has burst by time $t+s$? What is the limit of this probability as s approaches infinity?

Solve. 设泡沫在第 i 期破裂的概率为 P_i.

有两种理解方式, 第一种是恰在第 $t+s$ 期泡沫破裂, 我们有

$$P_{t+s} = 1 - \alpha^s.$$

当 s 趋于无穷, 泡沫破裂概率趋于 1

$$\lim_{s \to \infty} P_{t+s} = 1.$$

第二种是在第 $t+s$ 期及以前泡沫破裂, 我们有

$$P = \sum_{i=1}^{s} P_{t+i} = (1-\alpha)\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{s-1}\right).$$

当 $s \to \infty$, 我们有

$$P = \sum_{i=1}^{\infty} P_{t+i} = (1-\alpha)\sum_{i=1}^{\infty} \alpha^{i-1} = (1-\alpha)\frac{1}{1-\alpha} = 1.$$

□

3.3 Intrinsic Bubbles

Suppose that dividends follow a random walk: $D_t = D_{t-1} + e_t$, where e is white noise.

3.3.1 No Bubbles Price

In the absence of bubbles, what is the price of the stock in period t?

Solve. 没有泡沫时, t 时刻的资产价格

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t\left[\frac{D_{t+s}}{(1+r)^s}\right].$$

由于股息随机，所以 $\forall s > 0$, t 时刻的预期 $\mathbb{E}_t D_{t+s} = D_t$. 我们有

$$P_t = \sum_{s=1}^{\infty} \frac{D_t}{(1+r)^s} = D_t \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} = \frac{D_t}{r}.$$

□

3.3.2 First-Order Condition with Intrinsic Bubbles

Suppose that P_t equals the expression derived in (i) plus b_t, where $b_t = (1+r)b_{t-1} + ce_t$, $c > 0$. Is consumers' first-order condition derived in part (a) of Problem 2 still satisfied? In what sense do stock prices overreact to changes in dividends?

Solve. 由题设, 资产价格

$$P_t = \frac{D_t}{r} + b_t = \frac{D_t}{r} + (1+r)b_{t-1} + ce_t.$$

同样地, $t+1$ 期价格

$$P_{t+1} = \frac{D_{t+1}}{r} + (1+r)b_t + ce_{t+1} = \frac{D_t + e_{t+1}}{r} + (1+r)b_t + ce_{t+1}.$$

两边同时比上 $1+r$, 再取 t 期时的期望, 我们有

$$\mathbb{E}_t\left[\frac{P_{t+1}}{1+r}\right] = \frac{D_t}{r(1+r)} + b_t.$$

两边加上 $\mathbb{E}_t\left[\dfrac{D_{t+1}}{1+r}\right]$, 有

$$\mathbb{E}_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]=\mathbb{E}_t\left[\frac{D_{t+1}}{1+r}\right]+\frac{D_t}{r(1+r)}+b_t=\frac{(1+r)D_t}{r(1+r)}+b_t.$$

即

$$\mathbb{E}_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]=\mathbb{E}_t\left[\frac{D_{t+1}}{1+r}\right]+\frac{D_t}{r(1+r)}+b_t=\frac{(1+r)D_t}{r(1+r)}+b_t=\frac{D_t}{r}+b_t.$$

即得证满足原消费者的一阶条件. □