Notes of Intermediate Macroeconomics

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2024.09

Contents

1 Solow Growth Model

1.1 Households and Production

Assumptions

Closed economy, with a unique final good. Which means $NX = 0$. Capital is the same as the final good of the economy, but used in the production process of more goods.

Discrete time running to an infinite horizon. Time is indexed by $t = 0, 1, 2, \cdots, n$, where $n \to \infty$.

Economy is inhabited by a large number of households, and for now households will NOT be optimizing(最优化). Which is the main difference between the Solow Growth Model and the Neoclassical Growth Model.

All the households are identical(同一的). So the economy admits a representative household.

Households save a constant exogenous fraction s of their disposable income. Same assumption used in basic Keynesian Model and in the Harrod-Domer Model, at odds with reality.

All firms have access to the same production function. Economy admits a representative firm, with a representative (or aggregate) production function. Aggregate production function for the unique final good is

$$Y_t = F(K_t, L_t, A_t).$$

A_t is a shifter of the production function. Broad notion of technology. Technology is free. It is publicly available as a non-excludable, non-riavl good.

1.2 Mathematical Assumptions

Definition **[Homogeneous of Degree n]:** The function f is homogeneous of degree n on x and y, if and only if for $\forall \lambda \in \mathbb{R}$, we have

$$f(\lambda x, \lambda y, z) = \lambda^n f(x, y, z).$$

Theorem **[Euler's Theorem]:** If the function f is continuously differentiable on x and y, f_x and f_y are used to denote partial derivative of the function on x and y. x and y are of the same order n, then for $\forall x, y \in \mathbb{R}$, we have

$$nf(x, y, z) = f_x(x, y, z)x + f_y(x, y, z)y.$$

And, f_x , f_y itself are $n - 1$ times homogeneous on x and y.

Assumption 1 **[Continuity, Differentiability, Positive and Diminishing(递减) Marginal Products, and Constant Returns to Scale]** The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \triangleq \frac{\partial F(\cdot)}{\partial K} > 0, F_L(K, L, A) \triangleq \frac{\partial F(\cdot)}{\partial L} > 0$$

$$F_{KK}(K,L,A) \triangleq \frac{\partial F^2(\,\cdot\,)}{\partial K} < 0, F_{LL}(K,L,A) \triangleq \frac{\partial F^2(\,\cdot\,)}{\partial L} < 0$$

Moreover, F exhibits constant returns to scale in K and L. That is to say, F is linearly homogeneous (homogeneous of degree 1) in these two variables.

Assumption 2 [**Inada Conditions**] F satisfies the Inada Conditions

$$\lim_{K\to 0} F_K(\,\cdot\,) = \infty, \lim_{K\to\infty} F_K(\,\cdot\,) = 0,$$

for $\forall L > 0, \forall A$.

$$\lim_{L\to 0} F_L(\,\cdot\,) = \infty, \lim_{L\to\infty} F_L(\,\cdot\,) = 0,$$

for $\forall K > 0, \forall A$.

Important in ensuring the existence of interior equilibria.

1.3 Market Structure, Endowments and Market Clearing

Assumptions

Markets are competitive.

Households own all the labor, which they supply inelastically(无弹性地).

Endowment of labor in the economy, $\bar{L}_t$, and all of this will be supplied regardless of the price.

The labor market clearing condition can then be expressed as

$$L_t = \bar{L}_t.$$

For all t, where L_t denotes the demand for labor (and also the level of employment).

More generally, should be written in Complementary Slackness Form. In particular, let the wage rate at time t be w_t, then the labor market clearing condition takes the form

$$L_t \leqslant \bar{L}_t, w_t \geqslant 0, and (L_t - \bar{L}_t)w_t = 0.$$

But **Assumption 1** and **Competitive Labor Markets** make sure that wages have to be strictly positive, i.e. $w_t > 0$.

Households also own the capital stock of the economy and rent it to firms. Take initial holdings, K_0, as given.

Denote the rental price of capital at time t be R_t.

Capital market clearing condition

$$K_t^s = K_t^d.$$

Capital depreciates at the rate δ. Then, the interest rate faced by the household will be $r_t = R_t - \delta$.

1.4 Firm Optimization

Only need to consider the profit maximization problem (equivalently, cost minimization problem) of a representative firm

$$\max_{K_t \geq 0, L_t \geq 0} F(K_t, L_t, A_t) - R_t K_t - w_t L_t.$$

Since there are no irreversible(不可逆的) investments or costs of adjustments, the production side can be represented as s STATIC maximization problem.

Notice:

1. Problem is set up in terms of aggregate variables.
2. Nothing multiplying the F term, price of the final good has normalized(标准化) to 1.
3. Already imposes competitive factor markets: firm is taking as given w_t and R_t.

Since F is differentiable, first-order necessary conditions imply

$$w_t = F_L(K_t, L_t, A_t)$$

$$R_t = F_K(K_t, L_t, A_t).$$

We used K_t and L_t, the amount of capital and labor used by firms. Solving for K_t and L_t, we can derive the capital and labor demands of firms in this economy at rental prices R_t and w_t.

Proposition Suppose **Assumption 1** holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y_t = w_t L_t + R_t K_t.$$

Proof. From **Euler Theorem** for the case of $m = 1$, i.e. constant returns to scale. And from **First-order Condition**, we have

$$\begin{aligned} Y_t &= F(L_t, K_t) \\ &= F_L L_t + F_K K_t \\ &= w_t L_t + R_t K_t. \end{aligned}$$

Thus firm make no profits, so ownership of firms does not need to be specified(指定). □

1.5 Fundamental Law of Motion

Law of Motion for Capital:

Recall that K depreciates exponentially(指数地) at the rate of δ, so

$$K_{t+1} = (1 - \delta) K_t + I_t.$$

where I_t is the investment at time t.

From national income accounting for a closed economy,

$$Y_t = C_t + I_t.$$

Behavioral rule of constant saving rate simplifies the structure of equilibrium considerably. Since the economy is closed(and there is no government spending)

$$S_t = I_t = Y_t - C_t.$$

Individuals are assumed to save a constant fraction s of their income

$$S_t = sY_t,$$

$$C_t = (1-s)Y_t.$$

Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s_{t+1} = (1-\delta)K_t + sY_t$$

Market Clearing Conditions:
Setting supply and demand equal to each other, this implies

$$K^s_t = K_t.$$

We also have

$$L_t = \bar{L}_t.$$

Combining **Market Clearing Conditions** with **Law of Motion for Capital** and the production function, we have

$$K^s_{t+1} = (1-\delta)K_t + sF(K_t, L_t, A_t).$$

Equilibrium of the Solow Growth Model is described by this equation together with laws of motion for L_t and A_t.

1.6 Equilibrium in Discrete-Time

Solow Growth Model is a mixture of an old-style Keynesian Model and a modern dynamic macroeconomic model. Households do NOT optimize, but firms still maximize and factor markets clear.

In basic Solow Growth Model for a given sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K_t, Y_t, C_t, w_t, R_t\}_{t=0}^{\infty}$ such that K_t satisfies

$$K_{t+1} \leq (1-\delta)K_t + F(K_t, L_t, A_t) - C_t.$$

Y_t is given by $Y_t = F(K_t, L_t, A_t)$; C_t is given by $C_t = (1-s)Y_t$; w_t is given by $w_t = F_L(K_t, L_t, A_t)$; R_t is given by $R_t = F_K(K_t, L_t, A_t)$.

Note an equilibrium is defined as an entire path of allocations(分配) and prices, NOT a static object.

1.6.1 Equilibrium Without Population Growth and Technological Progress

<u>**Assumptions**</u>

There is no population growth. Total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L_t = L$.

No technological progress. So that $A_t = A$.

Define the capital-labor ratio of the economy as

$$k_t \triangleq \frac{K_t}{L}.$$

Using the constant returns to scale assumption, we can express output (income) pre capita

$$y_t \triangleq \frac{Y_t}{L}.$$

as

$$y_t \triangleq \frac{Y_t}{L} = F(\frac{K_t}{L}, 1, A) \triangleq f(k_t).$$

Note that $f(k)$ here depends on A, so it could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.

From **Euler Theorem**

$$R_t = f'(k_t)$$

and

$$\begin{aligned} w_t &= \frac{Y_t - R_t K_t}{L_t} \\ &= y_t - R_t k_t \\ &= f(k_t) - f'(k_t)k_t. \end{aligned}$$

(Remember that $Y_t = F(K_t, L_t, A_t) = w_t L_t + R_t K_t$.)

Both are positive from **Assumption 1**.

The per capita representation of the aggregate production function

$$k_{t+1} = (1-\delta)k_t + sf(k_t).$$

It can be referred to as the equilibrium difference equation of the Solow Model. The other equilibrium quantities can be obtained from the capital-labor ratio k_t.

Definition A steady state equilibrium without technological progress and population growth is an equilibrium path in which $k_t = k^\star$ for all t.

The economy will tend to this steady state equilibrium over time.

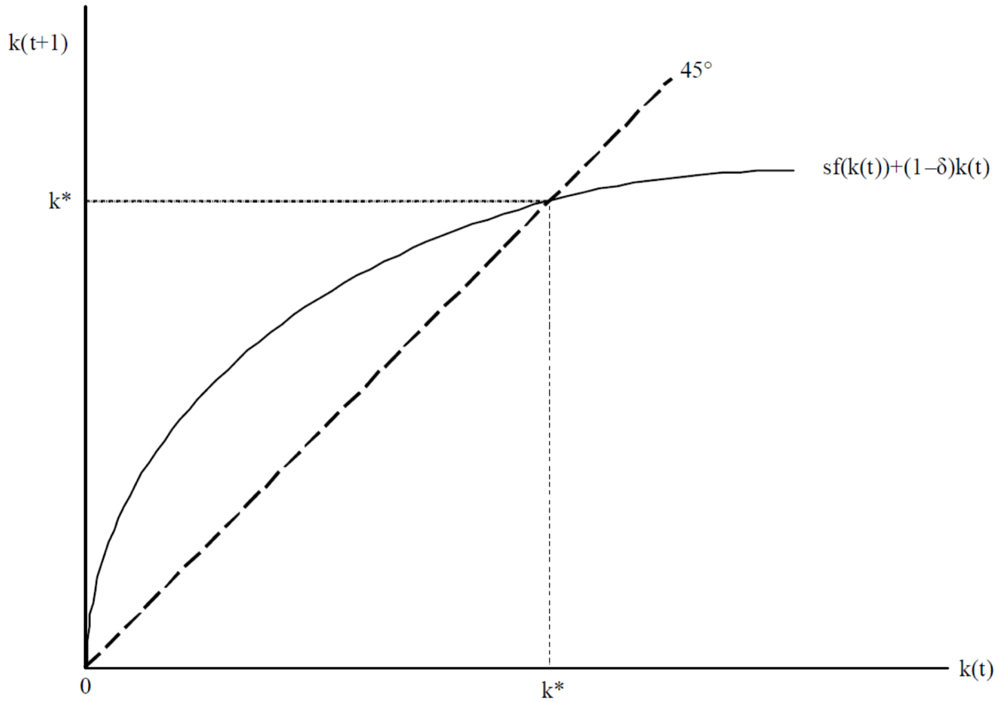

Figure 1 Steady State Capital-Labor Ratio

Thick curve represents the law of motion for capital per capita, the dashed line corresponds to the 45 degree line.

Their (positive) intersection gives the steady state value of the capital-labor ratio $k^\star$

$$k^\star = (1-\delta)k^\star + sf(k^\star)$$
$$\frac{f(k^\star)}{k^\star} = \frac{\delta}{s}.$$

Their is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$. Will ignore this intersection throughout:

If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium.

This intersection, even when it exists, is an unstable point. It has no economic interest for us.

Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. It depicts the levels of consumption and investment in a single figure and emphasizes the steady-steady equilibrium sets investment $sf(k)$ equal to the amount of capital that needs to be "replenished"(补充) δk.

Proposition Consider the basic Solow Growth Model and suppose that **Assumption 1** and **Assumption 2** hold. Then there exists a **Unique Steady State Equilibrium** where the capital-labor ratio $k^\star \in (0, \infty)$ is given by

$$\frac{f(k^\star)}{k^\star} = \frac{\delta}{s}.$$

Per capita output is given by

$$y^\star = f(k^\star).$$

And per capita consumption is given by

$$c^\star = (1-s)f(k^\star).$$

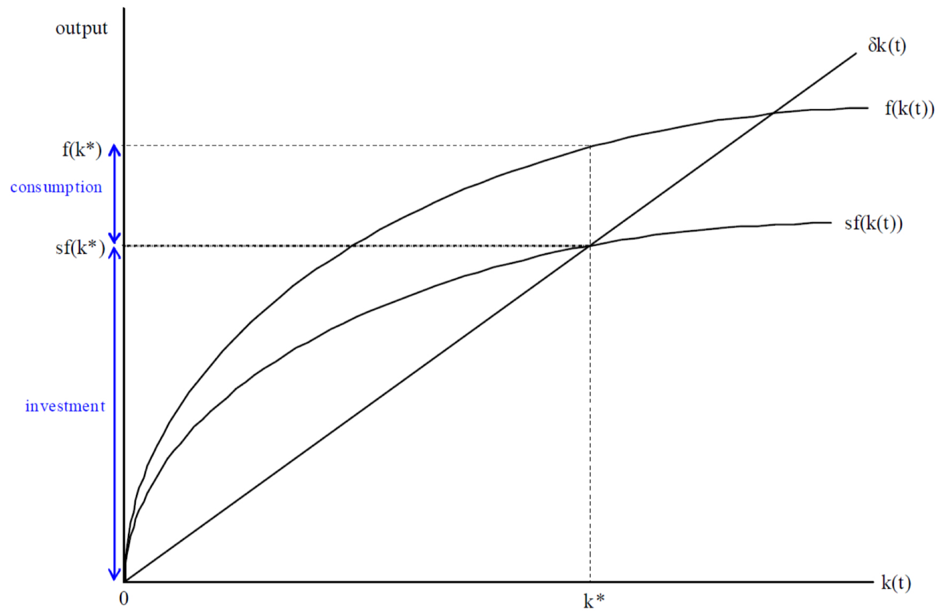

Figure 2 Consumption and Investment in Steady State

Proof. The preceding argument establishes that any $k^\star$ satisfied $\dfrac{f(k^\star)}{k^\star} = \dfrac{\delta}{s}$ is a steady state.

To establish existence, note that from **Assumption 2** and **L'Hospital's Rule**, $\lim_{k \to 0} f(k)/k = \infty$ and $\lim_{k \to \infty} f(k)/k = 0$.

Moreover, $f(k)/k$ is continuous from **Assumption 1**, so by the **Intermediate Value Theorem** there exists $k^\star$ such that $\dfrac{f(k^\star)}{k^\star} = \dfrac{\delta}{s}$ is satisfied.

To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial f(k)/k}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0.$$

Where the last equality uses

$$w_t = f(k_t) - f'(k_t)k_t > 0.$$

Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value $k^\star$ that satisfies $\dfrac{f(k^\star)}{k^\star} = \dfrac{\delta}{s}$. □

Comparative statics(比较静态) with respect to s and δ are straightforward for $k^\star$ and $y^\star$. But $c^\star$ will not be monotone(单调的) in the saving rate (e.g. $s = 1$).

1.6.2 Golden Rule

In fact, there will exist a specific level of the saving rate, s_{gold} referred to as the **Golden Rule** saving rate, which maximizes $c^\star$. But can not say whether the golden rule saving rate is **better** than some other saving rate.

The steady state relationship between c^* and s and suppress(抑制) other parameters(参数)

$$c^*(s) = (1-s)f(k^*(s)) = f(k^*(s)) - \delta k^*(s).$$

The equality exploits(利用) that in steady state $sf(k) = \delta k$.

Differentiating with respect to s

$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta]\frac{\partial k^*(s)}{\partial s}.$$

s_{gold} is such that $\dfrac{\partial c^*(s_{gold})}{\partial s} = 0$. The corresponding steady state golden rule capital stock is defined as k_{gold}.

Proposition In the basic Solow Growth Model, the highest level of steady state consumption is reached for s_{gold}, with the corresponding steady state capital level k_{gold} such that

$$f'(k^*_{gold}(s)) = \delta.$$

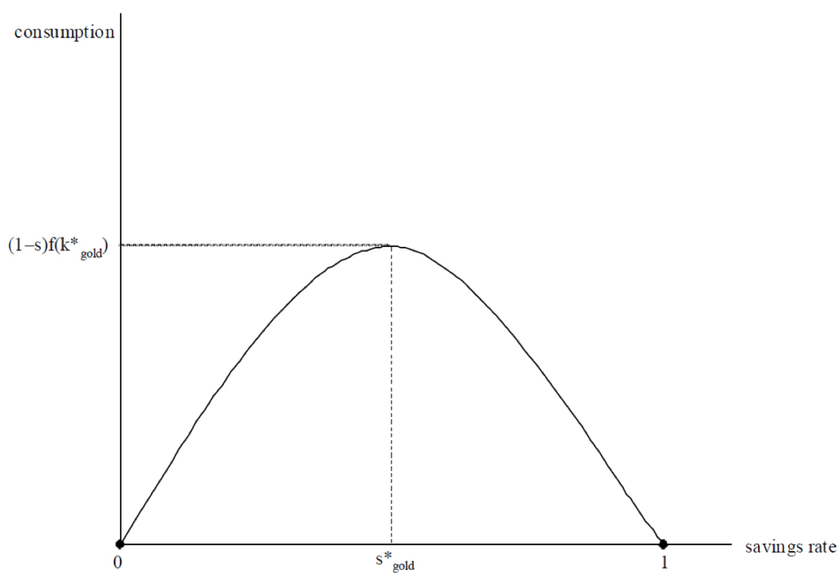

Figure 3 The Golden Rule

When the economy is below k_{gold}, higher saving will increase consumption; when it is above k_{gold}, steady state consumption can be increased by saving less. In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough **(dynamic ineffiency)**.

Such dynamic ineffiency will not arise once we endogenize consumption-saving decisions.

1.6.3 Discrete-Time Solow Model Redux

Per capita capital stock evolves according to

$$k_{t+1} = sf(k_t) = (1-\delta)k_t.$$

The steady state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

Consumption is given by

$$C_t = (1-s)Y_t.$$

And factor prices are given by

$$\begin{aligned} R_t &= f'(k_t) > 0. \\ w_t &= f(k_t) - f'(k_t)k_t > 0. \end{aligned}$$

1.6.4 Transitional Dynamics

Definition [**Equilibrium Path**] Not simply steady state, but entire path of capital stock, output, consumption and factor prices.

In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus the steady state equilibrium; In economics, non-steady state behavior also governed by optimizing behavior of households and firms and market clearing.

The **Transitional Dynamics** of the equilibrium difference equation $k_{t+1} = (1-\delta)k_t + sf(k_t)$ starting from an arbitrary initial capital-labor ratio $k_0 > 0$.

Key Question: Whether economy will tend to steady state and how it will behave along the transition path?

Proposition Suppose that **Assumption 1** and **Assumption 2** hold, then the steady state equilibrium of the Solow Growth Model described by the difference equation $k_{t+1} = (1-\delta)k_t + sf(k_t)$ is globally asmptotically(渐进地) stable, and starting from any $k_0 > 0$, k_t monotonically converges to k^*.

Proof. Let $g(k) \triangleq sf(k) + (1-\delta)k$. First observe that $g'(k) > 0$ for all k.

Next from the equation

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + sf(k_t), \\ k_{t+1} &= g(k_t). \end{aligned}$$

with a unique steady state at k^*.

From $\dfrac{f(k^*)}{k^*} = \dfrac{\delta}{s}$, the steady state capital k^* satisfies $\delta k^* = sf(k^*)$, or $k^* = g(k^*)$.

Recall that $f(\,\cdot\,)$ is concave and differentiable form **Assumption 1** and satisfies $f(0) > 0$ from **Assumption 2**.

(See **Concave Function** and **Convex Function** from wiki.)

For any strictly concave differentiable function

$$f(k) > f(0) + kf'(k) \geq kf'(k).$$

The second inequality uses the fact that $f(0) \geq 0$.

The above equation implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf(k^*) + 1 - \delta < 1$. Therefore

$$g'(k^*) \in (0, 1).$$

The simple result then establishes local asymptotic stability.

To prove global stability, note that for all $k_t \in (0, k^*)$

$$k_{t+1} - k^* = g(k_t) - g(k^*) = -\int_{k_t}^{k^*} g'(k)\mathrm{d}k < 0.$$

Second line uses the **Fundamental Theorem of Calculus**, and third line follows from the observation that $g' > 0$ for all k.

Next, the **Law of Motion** for per capita capital also implies

$$\frac{k_{t+1} - k_t}{k_t} = s\frac{f(k_t)}{k_t} - \delta > s\frac{f(k^*)}{k^*} - \delta = 0.$$

Moreover, for any $k_t \in (0, k^* - \varepsilon)$, this is uniformly so.

Second line uses the fact that $f(k)/k$ is decreasing in k and last line uses the definition of k^*.

These two arguments together establish that for all $k_t \in (0, k^*), k_{t+1} \in (k_t, k^*)$.

Therefore, $\{k_t\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable. $\square$

Stability result can be seen diagrammatically in the Figure 4. Starting from initial capital stock $k_0 < k^*$, economy grows towards k^*, capital deepening and growth of per capita income. If the economy were to start with $k_0 > k^*$, reach the steady state by decumulating capital and contracting.

As a consequence:

Proposition Suppose that **Assumption 1** and **Assumption 2** hold, and $k_0 < k_*$, then $\{w_t\}_{t=0}^{\infty}$ is an increasing sequence and $\{R_t\}_{t=0}^{\infty}$ is a decreasing sequence. If $k_0 > k^*$, the opposite results apply.

Thus far Solow Growth Model has a number of nice properties but no growth, except when the economy starts with $k_0 < k^*$.

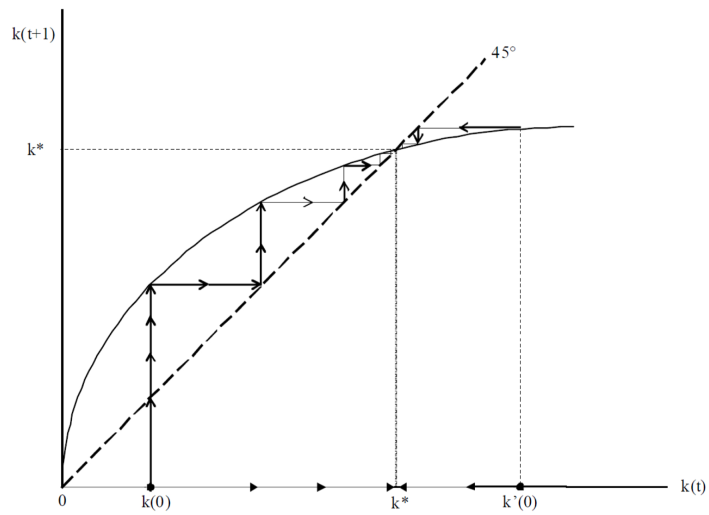

Figure 4 Transitional Dynamics in basic Solow Model

1.6.5 *From Difference to Differential Equations

Start with a simple difference equation

$$x_{t+1} - x_t = g(x_t).$$

Now consider the following approximation for any $\Delta t \in [0, 1]$

$$x_{t+\Delta t} - x_t \approx \Delta t g(x_t).$$

When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives the first equation. In-between it is a linear approximation, not too bad if

$$g(x) \approx g(x_t), \forall x \in [x_t, x_{t+1}].$$

Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{\Delta t} = \dot{x}_t \approx g(x_t).$$

where

$$\dot{x}_t \triangleq \frac{dx_t}{dt}.$$

This equation is a differential equation representing similar equation for the case in which t and $t + 1$ is **small**.

1.7 Equilibrium in Continuous-Time

1.7.1 Equilibrium with Population Growth

Nothing has changed on the production side, factor prices equations are as before, now interpreted as instantaneous(瞬间的) wage and rental rates.

Savings are again

$$S_t = sY_t.$$

Consumption equation is as before.

Introduce population growth

$$L_t = \exp(nt)L_0.$$

Recall

$$k_t \triangleq \frac{K_t}{L_t}.$$

Implies

$$\dot{k}_t = \frac{\dot{K}_t}{L_t} - \frac{\dot{L}_t}{L_t^2}K_t.$$

Divided by k_t

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} - n.$$

From previous defining instantaneous change in a variable

$$\begin{aligned}
\dot{K}_t &= sF(K_t, L_t, A_t) - \delta K_t \\
\frac{\dot{K}_t}{L_t} &= sf(k_t) - \delta\frac{K_t}{L_t} \\
\frac{\dot{K}_t}{K_t} &= \frac{sf(k_t)}{k_t} - \delta.
\end{aligned}$$

Recall the last equation from previous

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - n = \frac{sf(k_t)}{k_t} - (\delta + n).$$

Definition In the basic Solow Model in continuous time with population growth at rate n, NO technological progress and an initial capital stock K_0, an **Equilibrium Path** is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K_t, L_t, Y_t, C_t, w_t, R_t]_{t=0}^{\infty}$ such that L_t satisfies $L_t = \exp(nt)L_0$, $k_t \triangleq \frac{K_t}{L_t}$ satisfies $\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n)$, Y_t is given by the aggregate production function, C_t is given by $C_t = (1-s)Y_t$, and w_t and R_t are given by $R_t = f'(k_t)$ and $w_t = f(k_t) - f'(k_t)k_t$.

As before, steady state equilibrium involves k_t remaining constant at some level k^*.

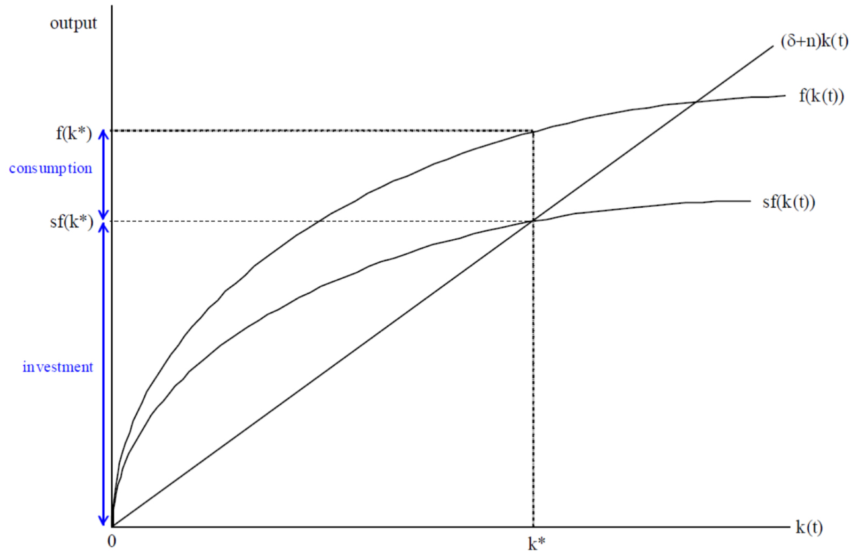

Figure 5 Steady State with Population Growth

1.7.2 Equilibrium Path in Continuous Time

Equilibrium path $\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n)$ has a unique steady state at k^*, which is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s}.$$

Proposition Consider the basic Solow Growth Model in continuous time and suppose that **Assumption 1** and **Assumption 2** hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by the above equation, per capita output and per capita consumption are given accordingly.

Similar comparative statics to the discrete time model.

1.7.3 Equilibrium with Technological Progress

Production function must admit representation of the form

$$Y_t = F(K_t, A_t L_t).$$

Moreover,suppose

$$\frac{\dot{A}_t}{A_t} = g$$
$$\frac{\dot{L}_t}{L_t} = n.$$

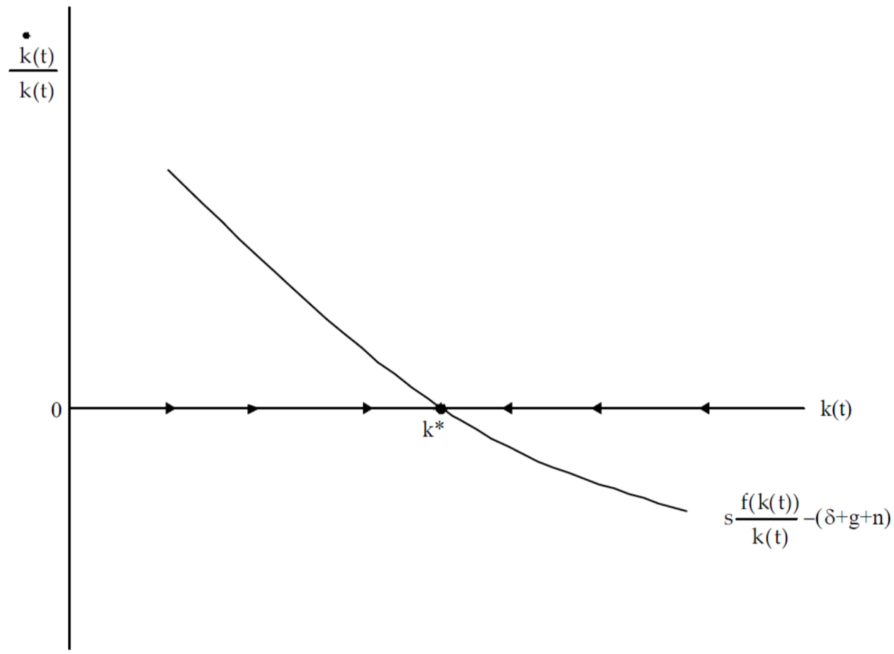

Figure 6 Converge to Steady State

Again using the constant saving rate

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

Now define $\hat{k}_t$ as the effective capital-labor ratio, i.e.

$$\hat{k}_t \triangleq \frac{K_t}{A_t L_t}.$$

Differentiating this expression with respect to time

$$\frac{\dot{\hat{k}}_t}{k_t} = \frac{\dot{K}_t}{K_t} - n - g.$$

Output per unit of effective labor can be written as

$$\hat{y}_t \triangleq \frac{Y_t}{A_t L_t} = F(\frac{K_t}{A_t L_t}, 1) \triangleq f(k_t).$$

Income per capita is

$$y_t = A_t \hat{y}_t = A_t f(k_t).$$

Clearly if y^t is constant, income per capita y_t will grow over time, since A_t is growing. Thus should not look for **steady state** where income per capita is constant, but for balanced growth paths, where income per capita grows at a constant rate.

Some transformed variables such as $\hat{y}_t$ or k_t remain constant. Thus balanced growth paths can be thought of as steady states of transformed model. Hence use the terms **Steady State** and balanced growth path interchangeably.

Substituting for $\dot{k}_t$

$$\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n + g).$$

Only difference is the presence of g: k is no longer the capital-labor ratio but the effective capital-labor ratio.

Proposition Consider the basic Solow Growth Model in continuous time, with Harrod-Neutral technological progress at the rate g and population growth at the rate n. Suppose the **Assumption 1** and **Assumption 2** hold, and define the effective capital-labor ratio as before. Then there exists a unique steady state (**Balanced Growth Path**) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}.$$

Per capita output and consumption grow at the rate g.

Equation $\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}$, emphasizes that now total savings, $sf(k)$, are used for replenishing(补充) the capital stock for three distinct reasons:

(1) Depreciation at the rate of δ.

(2) Population growth at the rate of n, which reduces capital per worker.

(3) Harrod-Neutral technological progress at the rate of g.

Now replrnishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

Proposition Suppose that **Assumption 1** and **Assumption 2** hold, then the Solow Growth Model with Harrod-Neutral technological progress and population growth in continuous time is asymptotically stable. i.e. Starting from any $k_0 > 0$, the effective capital-labor ratio converges to a steady state value $k^* (k_t \to k^*)$.

Now model generates growth in output per capita, but entirely exogenously.

1.7.4 Balanced Growth Path

Definition **[Balanced Growth Path]** A balanced growth path is an equilibrium where capital, output, consumption, wages, and return to capital grow at a constant rate.

This is roughly the case in many developed economics.

Solow capital accumulation $\dot{K} = sK^\alpha (AL)^{1-\alpha} - \delta K$.

$$g_K = \frac{\dot{K}}{K} = sK^{\alpha-1}(AL)^{1-\alpha} - \delta = s\left(\frac{AL}{K}\right)^{1-\alpha} - \delta.$$

On a balanced growth path, the growth rate of capital is constant. So AL/K must be constant.

$$g_{\frac{AL}{K}} = 0 \Rightarrow g_{AL} = g_K \Rightarrow g_A + g_L = g_K = g + n.$$

With production function $Y = (AL)^{1-\alpha} K^\alpha$, We have $\frac{Y}{K} = \left(\frac{AL}{K}\right)^{1-\alpha}$.

$$g_{\frac{Y}{K}} = g_{\left(\frac{AL}{K}\right)^{1-\alpha}} = 0 \Rightarrow g_Y = g_K.$$

We assume a constant saving rate $C = (1-s)Y$.

$$g_C = g_{(1-s)Y} \Rightarrow g_C = g_K = g + n.$$

$$W = \left(\frac{K}{L}\right)^{\alpha} A^{1-\alpha} = \left(\frac{K}{AL}\right)^{\alpha} A \Rightarrow g_W = g_{\left(\frac{K}{AL}\right)^{\alpha}} + g_A = g.$$

$$r = \alpha \left(\frac{AL}{K}\right)^{1-\alpha} \Rightarrow g_r = g_{\alpha\left(\frac{AL}{K}\right)^{1-\alpha}} = 0.$$

1.7.5 Comparative Dynamics

Comparative Dynamics: dynamic response of an economy to a change in its parameters(参数) or to shocks.

Different from comparative statics in Propositions above in that we interested in the entire path of adjustment of the economy following the stock or changing parameter. For brevity(简洁), we will focus on the continuous time economy.

One-time, unanticipated, permanent, increase in the saving rate from s to s'.

(1) Shifts curve to the right as shown by the dotted line, with a new intersection with horizontal axis, k^{**}.

(2) Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.

(3) Immediately, the capital stock remains unchanged (since it is a state variable).

(4) After this point, it follows the dashed arrows on the horizontal axis.

s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t''$.

(1) Starting at t', the economy follows the rightwards arrows until t'.

(2) After t'', the original steady state of the differential equation applies and leftwards arrows become effective.

(3) From t'' on wards, economy gradually returns back to its original balanced growth equilibrium k^*.

1.8 Conclusions

Simple and tractable(易处理的) framework, which allows us to discuss capital accumulation and the implications of technological progress.

Solow Model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.

Generate per capita output growth, but only exogenously: technological progress is a blackbox.

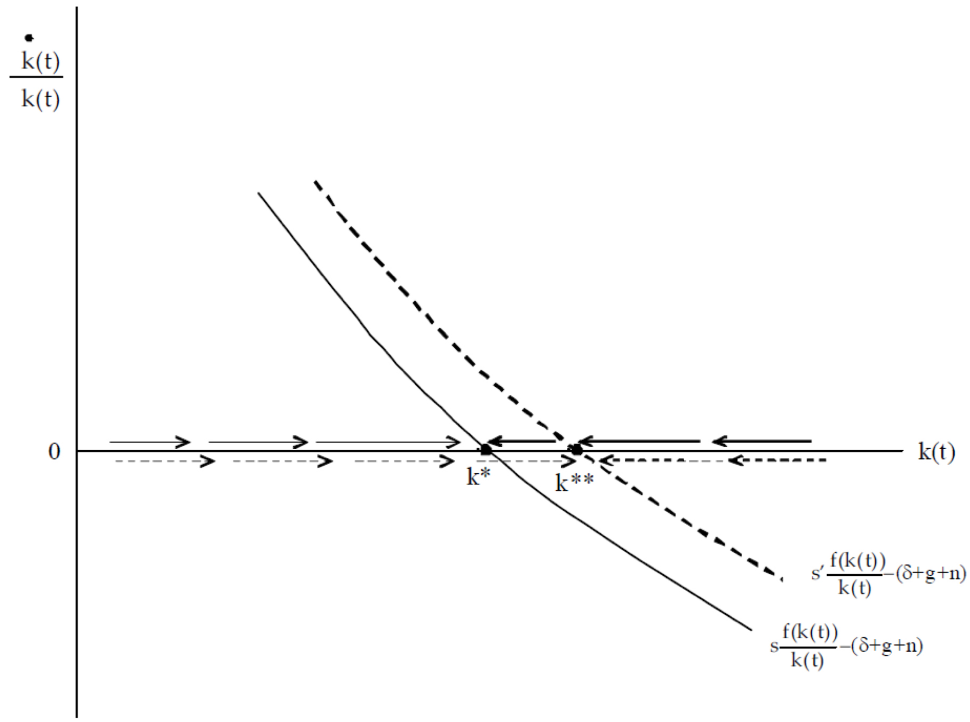

Figure 7 Dynamics following an increase in savings rate from s to s'

Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.

Need to dig deeper and understand what lies in these black boxes.

2 Growth Accounting

2.1 Basics

$$Y = F(K, AL)$$

How much of growth is **due to** growth in inputs(capital, labor, etc.) and growth in technology (A)?

Starting point

$$Y(t) = F[K(t), A(t)L(t)].$$

Differentiate with respect to time

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)}\dot{K}(t) + \frac{\partial Y(t)}{\partial L(t)}\dot{L}(t) + \frac{\partial Y(t)}{\partial A(t)}\dot{A}(t).$$

where $\frac{\partial Y}{\partial L}$ donates $\frac{\partial Y}{\partial AL}A$ and $\frac{\partial Y}{\partial A}$ donates $\frac{\partial Y}{\partial AL}L$.

Divide both sides by $Y(t)$

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{K(t)}{Y(t)}\frac{\partial Y(t)}{\partial K(t)}\frac{\dot{K}(t)}{K(t)} + \frac{L(t)}{Y(t)}\frac{\partial Y(t)}{\partial L(t)}\frac{\dot{L}(t)}{L(t)} + \frac{A(t)}{Y(t)}\frac{\partial Y(t)}{\partial A(t)}\frac{\dot{A}(t)}{A(t)}.$$

Elasticity of output with respect to capital and labor

$$\alpha_K(t) \triangleq \frac{K(t)}{Y(t)}\frac{\partial Y(t)}{\partial K(t)} \qquad \alpha_L(t) \triangleq \frac{L(t)}{Y(t)}\frac{\partial Y(t)}{\partial L(t)}.$$

In fact, $\frac{\partial Y(t)}{\partial K(t)} = MPK$, $\frac{\partial Y(t)}{\partial L(t)} = MPL$. $\alpha_K(t)$ is so called Capital Share, and $\alpha_L(t)$ is Labor Share.

We get

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha_K(t)\frac{\dot{K}(t)}{K(t)} + \alpha_L(t)\frac{\dot{L}(t)}{L(t)} + R(t).$$

Where

$$R(t) = \frac{A(t)}{Y(t)}\frac{\partial Y(t)}{\partial A(t)}\frac{\dot{A}(t)}{A(t)}.$$

is referred as the Solow Residual. One perspective: measure of our ignorance.

In principle measurable: growth in output $\frac{\dot{Y}(t)}{Y(t)}$, growth in capital $\frac{\dot{K}(t)}{K(t)}$, growth in labor $\frac{\dot{L}(t)}{L(t)}$. Elasticity(弹性) of output with respect to capital: $\alpha_K(t)$, Elasticity of output with respect to capital: $\alpha_L(t)$.

2.2 Measurement

Capital

Ideally we could measure flow of services from capital.

In practice, measure stock and assume flow is proportional to stock.

Perpetual inventory method(永续盘存制):

$$K(t+1) = K(t) + I(t) - \delta K(t).$$

Start with some $K(0)$, measure $I(t)$ from National Income and Product Accounts, use estimates of δ.

Quality of Inputs

Simple measure of labor input: hours worked. But workers differ, e.g in education and health. Increase in output may be due to increases in labor quality.

Jorgenson and Griliches (1967): Disaggregate inputs by schooling, etc. Weight each category by average wage.

Growth in overall labor input weighted average of categories, can also be done for capital.

If labor and capital earn their marginal product

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} \qquad w(t) = \frac{\partial Y(t)}{\partial L(t)}.$$

In this case output elasticities become factor shares

$$\alpha_K(t) = \frac{r(t)K(t)}{Y(t)} \triangleq S_K(t) \qquad \alpha_L(t) = \frac{w(t)L(t)}{Y(t)} \triangleq S_L(t).$$

Data on factor shares usually used to estimate $\alpha_K(t)$ and $\alpha_L(t)$. But this is only valid under idealized assumptions. (i.e. perfect competition)

2.3 Estimate

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha_K(t)\frac{\dot{K}(t)}{K(t)} + \alpha_{L(t)}\frac{\dot{L}(t)}{L(t)} + R(t).$$

Alternative approach: Estimate this equation using data on $\dot{Y}(t)/Y(t), \dot{Y}/K(t), \dot{Y}/L(t)$. Recover $\alpha_K(t)$ and $\alpha_L(t)$ as parameters, recover $R(t)$ as a residual.

Why not do this instead?

Would be hard since productivity affects inputs. (i.e. labor and capital are endogenous)

2.3.1 Dual Growth Accounting: Hsieh

We start with the accounting identity

$$Y = rK + wL.$$

Take logarithms and differentiate with respect to time

$$\frac{\dot{Y}}{Y} = S_K(\frac{\dot{r}}{r} + \frac{\dot{K}}{K}) + S_L(\frac{\dot{w}}{w} + \frac{\dot{L}}{L}).$$

Rearrange

$$\chi_{TFP} = \frac{\dot{Y}}{Y} - S_K(\frac{\dot{K}}{K} + \frac{\dot{K}}{K}) + S_L(\frac{\dot{w}}{w} + \frac{\dot{L}}{L}).$$

LHS: “primal” measure of Solow residual (what we had before)

RHS: “dual” measure of Solow residual

Primal and dual approach should yield the same answer.If one is (in)valid, the other is (in)valid.

Hsieh applied dual approach to East Aisa **Tigers**.

Hsieh argues:

1. NIPA data implies that capital/output ratio rose sharply.

2. Since factor shares are roughly constant, this implies that rate of return on capital should have fallen sharply.

3. True for Korea but not for Singapore. 4.Singapore's NIPA overstate investment.

2.3.2 Growth Accounting and Sources of Growth

Growth accounting is just accounting, not causal analysis.

e.g.,Suppose A and L are constant, x is labor-augmented growth in technology.

$$Y = AK^{\alpha}(Le^{xt})^{1-\alpha}.$$

Take logarithms and differentiate with respect to time

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)x.$$

In Solow and Ramsey models: capital-output ratio will be constant along a balanced growth path

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = x.$$

αx of growth attributed to growth of capital.

TFP growth measured to be $\hat{g} = (1-\alpha)x$.

To attribute to technology both direct and indirect effects on GDP, we need to divide measured TFP growth by $(1-\alpha)$.

2.3.3 Alternative Growth Accounting Approach

Start with a Cobb-Douglas production function

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}.$$

Here H_t denotes human capital.

Divide both sides by Y_t^α and raise to power $1/(1-\alpha)$

$$Y_t = \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} H_t Z_t.$$

where $Z_t = A_t^{\frac{1}{1-\alpha}}$.

Divide through buy L_t

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} \frac{H_t}{L_t} Z_t.$$

Decompose per capita(or per hour) growth into Capital deepening: K_t/Y_t, Growth in human capital per hour: H_t/L_t, Total factor productivity: Z_t.

Importantly Solow and Ramsey model imply that K_t/Y_t is constant along a balanced growth path.

Take logarithms and differentiate with respect to time to get a growth accounting equation. This approach popularized bu Klenow and Rodriguez-Clare.

2.3.4 Exponential Versus Linear

Central claim: We usually assume growth is exponential

$$A_{t+\tau} = A_t(1+g)^\tau.$$

In fact, growth is linear

$$A_{t+\tau} = A_t + b\tau.$$

Exponential growth:

$$\begin{aligned} A_{t+\tau} = A_t(1+g)^\tau &\Rightarrow \ln A_{t+\tau} = \ln A_t + \tau \ln(1+g) \\ &\Rightarrow \ln A_{t+\tau} \approx \ln A_t + g\tau. \end{aligned}$$

Linear growth:

$$A_{t+\tau} = A_t + b\tau.$$

Growth cannot have been linear forever. If so, $A(t)$ would be negative at some point in the past. Philippon propose that **General Purpose Technologies** cause breaks.

3 Consumption

3.1 A Two-Period Model of Consumption

3.1.1 Partial Equilibrium Case

Partial equilibrium: study problem of individual in isolation taking as given prices.

Two time periods $t = 1$ and $t = 2$.

Consumption c_1 and c_2, Income y_1 and y_2.

Utility function

$$u(c_1) + \beta u(c_2).$$

with u **strictly increasing, concave**, discount factor $0 < \beta < 1$.

Households solves

$$\begin{aligned}
&\max_{c_1,c_2,a} u(c_1) + \beta u(c_2). \\
\text{s.t.} \quad & c_1 + a = y_1 \\
& c_2 = y_2 + (1+r)a.
\end{aligned}$$

c_1, c_2 : consumption at $t = 1$ and $t = 2$. y_1, y_2 : income at $t = 1$ and $t = 2$. r : interest rate(for now exogenously given). a : saving(a can be negative).

Implicit assumption: can borrow and save as much you want at rate r.

Combine the budget constraints

$$\begin{aligned}
a &= \frac{c_2}{1+r} - \frac{y_2}{1+r} \\
c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r}.
\end{aligned}$$

Households problem is then

$$\begin{aligned}
&\max_{c_1,c_2} u(c_1) + \beta u(c_2) \\
\text{s.t.} \quad & c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.
\end{aligned}$$

Solve. Form a Lagrangian

$$\begin{aligned}
F(c_1, c_2, \lambda) &= u(c_1) + \beta u(c_2) + \lambda(c_1 + \frac{c_2}{1+r} - y_1 - \frac{y_2}{1+r}) \\
\frac{\partial F}{\partial c_1} &= u'(c_1) + \lambda = 0 \\
\frac{\partial F}{\partial c_2} &= \beta u'(c_2) + \frac{\lambda}{1+r} = 0
\end{aligned}$$

Eliminate λ, we have

Optimal condition

$$u'(c_1) = \beta(1+r)u'(c_2).$$

$\square$

A key equation named **Euler Equation** after Leonard Euler. An intertemporal optimality condition.

Related question: When to borrow/save? What if r increases?

3.1.2 General Equilibrium Case: One Type of Agents

Setup: log utility, two periods, bonds, bond price q_t, **exogenous** income Y_t, Y_{t+1} (more on the bond).

$$\begin{aligned}
\max_{C_t, B_t, C_{t+1}} \quad & \ln C_t + \beta \ln C_{t+1} \\
\text{s.t.} \quad & C_t + q_t B_t \le Y_t \\
& C_{t+1} \le Y_{t+1} + B_t.
\end{aligned}$$

Solve. Form a Lagrangian

$$\mathcal{L} = \ln C_t + \lambda_1(Y_t - C_t - q_t B_t) + \lambda_2(Y_{t+1} + B_t - C_{t+1}).$$

$B_t = 0$: Only one type of agent. $C_t = Y_t, C_{t+1} = Y_{t+1}, q_t = \beta \dfrac{Y_t}{Y_{t+1}}$.

Permanent income hypothesis does not work in this setting. The consumption smoothing channel is shut down for there is no essential saving tools in the model.

Rewrite $1 + r_t = \dfrac{1}{q_t}$

$$1 + r_t = \frac{Y_{t+1}}{\beta Y_t}.$$

$\square$

Fix bond supply at 0, an increase in Y_{t+1} reduces bond demand, drives down the bond price(increases the real rate t_t).

A Stochastic Case

Two states: A high state $Y_{t+1,1}$ with probability p, a low state $Y_{t+1,1}$ with probability $1-p$.

$$\begin{aligned}
\max_{C_t, B_t, C_{t+1,1}, C_{t+1,2}} \quad & \ln C_t + \beta p \ln C_{t+1,1} + \beta(1-p) \ln C_{t+1,2} \\
\text{s.t.} \quad & C_t + q_t B_t \le Y_t \\
& C_{t+1,1} \le Y_{t+1,1} + B_t \\
& C_{t+1,2} \le Y_{t+1,2} + B_t.
\end{aligned}$$

Solve. Form a Lagrangian

$$\begin{aligned}\mathcal{L} &= \ln C_t + \beta p \ln C_{t+1,1} + \beta(1-p)\ln C_{t+1,2} \\ &\quad + \lambda_1(Y_t - C_t - q_t B_t) + \lambda_2(Y_{t+1,1} + B_t - C_{t+1,1}) + \lambda_3(Y_{t+1,2} + B_t - C_{t+1,2}).\end{aligned}$$

FOCs:

$$\begin{aligned}\{C_t\}: &\qquad \frac{1}{C_t} = \lambda_1 \\ \{C_{t+1,1}\}: &\qquad p\beta\frac{1}{C_{t+1,1}} = \lambda_2 \\ \{C_{t+1,2}\}: &\qquad (1-p)\beta\frac{1}{C_{t+1,2}} = \lambda_3 \\ \{B_t\}: &\qquad q_t\lambda_1 = \lambda_2 + \lambda_3.\end{aligned}$$

We get

$$q_t\frac{1}{C_t} = \beta\left(p\frac{1}{C_{t+1,1}} + (1-p)\frac{1}{C_{t+1,2}}\right).$$

of which the right-hand side is the expected marginal utility of future consumption.

$$q_t\frac{1}{C_t} = \beta\mathbb{E}_t\frac{1}{C_{t+1}}.$$

□

Implications of Uncertainty: A mean-preserving increase in uncertainty will increase agents demand for smoothing(bond, saving). It will drives up the bond price, thus decrease r_t.

3.1.3 General Equilibrium Case: Two Types of Agents

They have **identical preferences**, but differ somehow in their endowments. Log utility, same size of two types of agents.

Endowment

$$\begin{aligned}(Y_{1,t}, Y_{1,t+1}) &= (1,0) \\ (Y_{2,t}, Y_{2,t+1}) &= (0,1)\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{C_{i,t}} = \beta\frac{1}{q_t}\frac{1}{C_{i,t+1}}, \quad i = 1,2 \\ &\Rightarrow \frac{C_{1,t+1}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{2,t}} \\ &\Rightarrow \frac{C_{2,t}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{1,t+1}}.\end{aligned}$$

Conditions

$$B_{1,t} + B_{2,t} = 0$$
$$C_{1,t} + C_{2,t} = Y_{1,t} + Y_{2,t} = 1$$
$$C_{1,t+1} + C_{2,t+1} = Y_{1,t+1} + Y_{2,t+1} = 1.$$

We have

$$\frac{1 - C_{2,t}}{C_{2,t}} = \frac{1 - C_{2,t+1}}{C_{2,t+1}}$$
$$C_{2,t} = C_{2,t+1}$$
$$C_{1,t} = C_{1,t+1}$$

Which means both agents **consume same** in period 1 and 2. However, we have $C_{1,t} > C_{2,t}$.
Plug back to the **Euler Equation** , get $q_t = \beta$.
Equilibrium

$$C_{i,t} + q_t C_{i,t+1} = Y_{i,t} + q_t Y_{i,t+1}$$
$$C_{i,t} = \frac{1}{1+\beta}(Y_{i,t} + \beta Y_{i,t+1})$$
$$C_{2,t} = C_{2,t+1} = \frac{\beta}{1+\beta}$$
$$C_{1,t} = C_{1,t+1} = \frac{1}{1+\beta}.$$

of which the first equation combines budget constraints for two periods.
Combining with $B_{i,t} = \frac{1}{q_t}(Y_{i,t} - C_{i,t})$.

$$B_{1,t} = \frac{1}{1+\beta} \qquad \text{(Saving)}$$
$$B_{2,t} = -\frac{1}{1+\beta} \qquad \text{(Borrow)}$$

Stochastic Heterogeneous Endowments

Setup: two future states, p probability of state 1, and $1-p$ probability of state 2.

$$Y_{1,t+1,1} + Y_{2,t+1,1} = 1$$
$$Y_{1,t+1,2} + Y_{2,t+1,2} = 1$$

of which the first 1 indicating agent and the last 1 indicating state.

Budget constraint at period 1:

$$C_{i,t} + q_{t,1}B_{i,t,1} + q_{t,2}B_{i,t,2} \leq Y_{i,t}.$$

Budget constraint at period 2:

$$\begin{aligned} C_{i,t+1,1} &\leq Y_{i,t+1,1} + B_{i,t,1} \\ C_{i,t+1,2} &\leq Y_{i,t+1,2} + B_{i,t,2} \end{aligned}$$

Two states: a high state $Y_{t+1,1}$ with probability p, a low state $Y_{t+1,2}$ with probability $1-p$.

$$\begin{aligned} \max_{C_{i,t},B_{i,t,1},B_{i,t,2},C_{i,t+1,1},C_{i,t+1,2}} \quad & \ln C_{i,t} + \beta p \ln C_{i,t+1,1} + \beta(1-p)\ln C_{i,t+1,2} \\ \text{s.t.} \qquad & C_{i,t} + q_{t,1}B_{i,t,1} + q_{t,2}B_{i,t,2} \leq Y_{i,t} \\ & C_{i,t+1,1} \leq Y_{i,t+1,1} + B_{i,t,1} \\ & C_{i,t+1,2} \leq Y_{i,t+1,2} + B_{i,t,2} \end{aligned}$$

Solve. Form a Lagrangian:

$$\begin{aligned} \mathcal{L} = & \ln C_{i,t} + \beta p \ln C_{i,t+1,1} + \beta(1-p)\ln C_{i,t+1,2} \\ & + \lambda_{i,1}(Y_{i,t} - C_{i,t} - q_{t,1}B_{i,t,1} - q_{t,2}B_{i,t,2}) \\ & + \lambda_{i,2}(Y_{i,t+1,1} + B_{i,t,1} - C_{i,t+1,1}) \\ & + \lambda_{i,3}(Y_{i,t+1,2} + B_{i,t,2} - C_{i,t+1,2}) \end{aligned}$$

FOCs:

$$\begin{aligned} \{C_{i,t}\}: \qquad & \frac{1}{C_{i,t}} = \lambda_{i,1} \\ \{C_{i,t+1,1}\}: \qquad & p\beta\frac{1}{C_{i,t+1,1}} = \lambda_{i,2} \\ \{C_{i,t+1,2}\}: \qquad & (1-p)\beta\frac{1}{C_{i,t+1,2}} = \lambda_{i,3} \\ \{B_{i,t,1}\}: \qquad & q_{t,1}\lambda_{i,1} = \lambda_{i,2} \\ \{B_{i,t,2}\}: \qquad & q_{t,2}\lambda_{i,1} = \lambda_{i,3} \end{aligned}$$

We get:

$$\begin{aligned} \frac{p}{1-p}\frac{C_{i,t+1,2}}{C_{i,t+1,1}} &= \frac{\lambda_{i,2}}{\lambda_{i,3}} \\ \frac{\lambda_{i,2}}{\lambda_{i,3}} &= \frac{q_{t,1}}{q_{t,2}} \end{aligned}$$

For agents:

$$\frac{C_{1,t+1,2}}{C_{1,t+1,1}} = \frac{C_{2,t+1,2}}{C_{2,t+1,1}}$$
$$\frac{C_{1,t+1,2}}{C_{2,t+1,2}} = \frac{C_{1,t+1,1}}{C_{2,t+1,1}}$$

$\square$

(Reference One agent,distribute for $t, t+1$.)

Household consumption problem is to maximize $\mathbb{E}_0 \sum_{t=0}^{T} \beta^t U(C_t)$.

$$C_t = \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} \mathrm{d}i\right]^{\frac{\theta}{\theta-1}}$$

subject to a budget constraint.

Problem can be divided into two parts: Solve for $c_t(i)$ subject to C_t, solve for C_t.

Expenditure on goods in budget constraint:

$$\int_0^1 p(i)c(i)\mathrm{d}i.$$

Define P_t as the minimum expenditure needed to purchase one unit of the composite consumption good C_t.

Then it turns out that

$$\int_0^1 p(i)c(i)\mathrm{d}i = P_t C_t.$$

So, we can write budget constraint without reference to $c(i)s$ and $p(i)s$. P_t is the **ideal price index**.

3.2 Keynes' Consumption Function

$$C_t = \alpha + \gamma(Y_t - T_t).$$

Consumption is a function of after-tax income. Marginal propensity(倾向) to consume(γ) between 0 to 1. **Interest rates not important.** Future income not important.

3.2.1 Keynesian Cross

Suppose I, G, NX are **exogenous**.(i.e, not functions of output directly or indirectly)

Planned expenditure (aggregate demand):

$$P\mathbb{E}_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t.$$

Suppose the output is completely **demand determined**. Output must equal $P\mathbb{E}_t$:

$$Y_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t.$$

A little algebra then yields:

$$Y_t = \frac{1}{1-\gamma}\left(\alpha - \gamma T_t + I_t + G_t + NX_t\right).$$

Government purchases multiplier(乘数) $= \dfrac{1}{1-\gamma}$.

Proof.

Government spends:ΔG(which raises income by ΔG)

First round change in consumption: $\gamma \Delta G$

Second round change in consumption: $\gamma^2 \Delta G$

Etc.

So, $\Delta Y_t = \Delta G_t + \gamma \Delta G_t + \cdots + \gamma^T \Delta G_t + \cdots = (1 + \gamma + \cdots + \gamma^T + \cdots)\Delta G_t = \dfrac{1}{1-\gamma}\Delta G_t.$ □

Tax cut multiplier $= \dfrac{\gamma}{1-\gamma}$.

Proof.

Tax cut:$-\Delta T$(which will not raises income)

First round change in consumption: $-\gamma \Delta T$(which will raises income by $\gamma \Delta T$)

Second round change in consumption: $-\gamma^2 \Delta T$

Etc.

So, $\Delta Y_t = -\gamma \Delta T_t - \cdots - \gamma^T \Delta T_t - \cdots = -(\gamma + \cdots + \gamma^T + \cdots)\Delta T_t = -\dfrac{1}{1-\gamma}\Delta T_t.$ □

Tax effect < Government Purchases effect.

Multiplier for change to **Autonomous Spending** (i.e.,α) same as for G.

Keynesian Cross is the simplest form in Keynesian Economics. It has **VERY** strong assumptions:

1. Simplistic consumption function.
2. Investment exogenous.
3. No price change as output changes(i.e.,Economy completely demand determined).
4. No monetary policy response(but wouldn't matter since nothing responds to interesting rate).

IS-LM Model: $I(r)$ + monetary policy response.

New Keynesian Model: **Modern** consumption function + Phillips curve + monetary policy.

The Problem of Thrift

Classical Economics:Saving is good, foundation for capital accumulation.

(Old) Keynesian Economics: Increased saving / fall in **autonomous** spending (i.e., α), thought to have contributed to causing the Great Depression. Widespread worry during WWII about **secular stagnation**. As people get richer, they will save larger share of income ($MPC < 1$), eventually too much saving, not enough demand, not enough investment opportunities.

Empirics of Consumption function

Early work looked at budget studies(i.e., cross section at a point in time)

$$\gamma = \Delta C/\Delta Y \approx 2/3.$$

Also analyzed aggregate saving over course of Great Depression. Savings rose as economy recovered.

3.3 Permanent Income Hypothesis and Life-Cycle Hypothesis

Originally developed independently by: Modigiani and Brumberg (1954) (Life-Cycle Hypothesis), Friedman (1957) (Permanent Income Hypothesis)

Basic idea:

1. Utility maximization and perfect markets imply that current consumption is determined by net present value of life-time income.

2. Dramatically different from Keynesian consumption function.

3.3.1 Households Consumption-Saving Problem

Assumptions

Known finite horizon T No uncertainty Constant interest rate

No durable goods (houses/cars/etc) Exogenous income process

Costless enforcement of contracts Natural borrowing limit

No bankruptcy (i.e., full commitment to repay debt)

Setup

Preferences:

$$\sum_{t=0}^{T} \beta^t U(C_t)$$

Savings/Borrowing technology: Household can save at rate r. Household can borrow at rate r uo to some limit. Household assets denoted A_t.

Initial assets: A_{t-1}.

Income stream: Y_t.

Household's Problem

$$\max_{C_1,\cdots,C_T} \sum_{t=0}^{T} \beta^t U(C_t)$$
$$\text{s.t.} \qquad \frac{A_t}{1+r} + C_t = Y_t + A_{t-1}.$$

The s.t. in this case is called **budget constraint**, but mathematically, this is not really a constraint, it is just a definition of A_t .

Budget Constraint

Real constraint in constraint on A_t sequence.

Simplest: **Natural** borrowing limit: $A_T \geq 0$.(i.e.,household cannot die with debt)

Alternative: No (unsecured) borrowing: $A_t > 0$.(much tighter/much more realistic)

Intertemporal Budget Constraint

With natural borrowing limit, sequence of one-period budget constraints can be consolidated into a single intertemporal budget constraint:

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}.$$

Present value of consumption cannot be larger than present value of income and assets.

This embeds the $A_T \geq 0$ constraint.

Household's Problem

$$\max_{C_1,\cdots,C_T} \sum_{t=0}^{T} \beta^t U(C_t)$$

$$\text{s.t.} \qquad \sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}.$$

Important to differentiate between: Choice variables: C_t (And A_t , for $t \geq 0$). Exogenous variables: A_{-1} and Y_t (and r and β).

Solve. One way to solve household's problem is to set up a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(C_t) - \lambda \left(\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} \right).$$

and derive Kuhn-Tucker conditions.

Differentiating FOC:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U'(C_t) - \frac{\lambda}{(1+r)^t} = 0 \Rightarrow \beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}.$$

The full set of optimality conditions additionally includes a complementary slackness condition

$$\lambda \left(\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} \right) = 0.$$

Notice from FOC that:

$$\lambda = \beta^t (1+r)^t U'(C_t).$$

If $U' > 0$,then $\lambda > 0$.

Implies that:

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} = 0.$$

Since $U' > 0$,intertemporal budget constraint holds with equality. Ofen we just impose this from the beginning.

$$\beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}$$
$$\beta^{t+1} U'(C_{t+1}) = \frac{\lambda}{(1+r)^{t+1}}$$

Divide one by the other

$$\frac{U'(C_t)}{\beta U'(C_{t+1})} = 1 + r.$$

Rearrange:

$$U'(C_t) = \beta(1+r)U'(C_{t+1}).$$

This equation is usually referred as the **consumption Euler equation**. □

Lagrangian math does not yield much intuition.

Solve. Alternative: Calculus of variations. We seek to

$$\max \ V(C) = \sum_{t=0}^{T} \beta^t U(C_t)$$
$$\text{s.t.} \qquad \sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \le A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}.$$

Here C denotes the sequence $\{C_0, C_1, \cdots, C_{T-1}, C_T\}$.

Suppose we have a candidate optimal path C_t^*. Let's consider a variation on this path: Save ε more at time t. Consume proceeds at time $t+1$.

Utility from new path:

$$V(C) = \cdots + \beta^t U(C_t^* - \varepsilon) + \beta^{t+1} U(C_{t+1}^* + \varepsilon(1+r)) + \cdots$$

If C_t^* is the optimum, then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0.$$

At the optimum, benefit of small variation must be 0.

$$\frac{\mathrm{d}V}{\mathrm{d}\varepsilon} = -\beta^t U'(C_t^* - \varepsilon) + (1+r)\beta^{t+1} U'(C_{t+1}^* + \varepsilon(1+r))$$
$$\left.\frac{\mathrm{d}V}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = -\beta^t U'(C_t^* - \varepsilon) + (1+r)\beta^{t+1} U'(C_{t+1}^*) = 0$$
$$U'(C_t) = \beta(1+r)U'(C_{t+1}).$$

□

The generic FOC in calculus of variations is called the **Euler equation** (or Euler-Lagrange equation). This is where the consumption Euler equation gets its name.

3.3.2 Permanent Income Hypothesis

Suppose $U(C_t) = \ln C_t$,then

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

Becomes

$$\frac{C_{t+1}}{C_t} = \beta(1+r).$$

The consumption growth $\frac{C_{t+1}}{C_t}$ does **not** depend on income growth $\frac{Y_{t+1}}{Y_t}$.

Suppose for simplicity that $\beta(1+r) = 1$. Consumption Euler equation becomes

$$U'(C_t) = U'(C_{t+1}).$$

which implies

$$C_t = C_{t+1}.$$

Consumers optimally smooth their consumption. Variation in consumption only due to: Variation in interest rates. Variation in marginal utility $U'(C_t)$ (e.g.,children,health)

Let's plug $C_t = C_{t+1}$ into intertemporal budget constraint

$$\begin{aligned} C_0 =& \Phi(r)\left(A_{-1} + \sum_{t=0}^{T}\frac{Y_t}{(1+r)^t}\right) \\ \Phi(r) =& \frac{1-\frac{1}{1+r}}{1-\left(\frac{1}{1+r}\right)^{T+1}} \end{aligned}$$

Consumption a function of present value of life-time income. Current income is not special.

Marginal propensity to consume(MPC): How much of a windfall extra dollar a household spends over some period time. The Permanent Income Hypothesis imply that MPC out of windfall gain is equal to $\Phi(r)$.

Suppose $T = 40$ and $r = 0.02$, $\Phi(r) = 0.035$.(Compare to Keynesian: $\Phi(r) = 0.667$)

3.3.3 *Infinite Horizon and Uncertainty

Let's consider a version of the household problem with:

Infinite horizon   Uncertainty   Heterogeneous(偏好) preferences

We still maintain:

No durable goods (houses/cars/etc)   Exogenous income process

Costless enforcement of contracts   No bankruptcy (i.e., full commitment to repay debt)

Natural borrowing limit

Why infinite horizon?

1. Altruism(利他主义): We love our children.

If we value our children's consumption like our own, intergenerational discounting is the same as intragenerational discounting.

If we however value giving (not children's consumption) things are different(warm glow bequests).

2. Simplicity:

Infinite horizon makes problem more stationary(固定).

In finite horizon problem, horizon is a state variable(i.e., affects optimal choice).

Solution to problem with long horizon similar to one with infinite horizon.

Nature of Uncertainty

Household i faces uncertainty about future income Y_{it+j} (include i to emphasize that risk is partly idiosyncratic).

Heterogeneity in income across households potentially yields heterogeneity in consumption C_{it}.

Why Assets are Traded?

If households are risk averse(厌恶), they will want to **buy insurance** against income risk.

Whether they can depends on what assets are traded.

Two polar cases:

1.Complete markets: Complete set of state contingent assets available.

2.Bonds only: Only non-state contingent asset available.

We will start by considering the complete markets case.

Natural Borrowing Limit

Household can **borrow**(sell assets) up to the point where it can repay for sure in all states of the world.

Rules out **Ponzi schemes**

Sell asset at time t.

Sell more assets at time $t+1$ to pay off interest/principle coming due.

Keep doing this ad infinitum.

Natural borrowing limit can be quite **tight**: If non-zero probability of zero future income, natural borrowing limit is zero.

Household's Problem

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i(C_{it}(s^t))$$

$$\text{s.t.} \qquad C_{it}(s^t) + \sum_{s^{t+1}|s^t} Q_t(s^{t+1}) A_{it}(s^{t+1}) = Y_{it}(s^t) + A_{i,t-1}(s^t).$$

a No Ponzi scheme constraint, and given $A_{i,-1}$.

The choice variables at time t are $C_{it}(s^t)$ and $A_{it}(s^{t+1})$.

$s^t = [s_1, s_2, \cdots, s_t]$ donates history of states up to date t.

$Q_t(s^{t+1})$ donates the time t price of Arrow security that pays of one unit of consumption in state s^{t+1}.

$A_{it}(s^{t+1})$ denotes quantity of Arrow security that pays off in state s^{t+1} that is purchased at time t by household i.

Solve.

$$\begin{aligned}\mathcal{L}_t = \mathbb{E}_0 \sum_{j=0}^{\infty} \beta^j (U_i(C_{i,t+j}(s^{t+j})) - \lambda_{i,t+j}(s^{t+j})(C_{i,t+j}(s^{t+j}) \\ + \sum_{s^{t+j+1}|s^{t+j}} Q_{t+j}(s^{t+j+1})A_{i,t+j}(s^{t+j+1}) - Y_{i,t+j}(s^{t+j}) + A_{i,t+j-1}(s^{t+j}))).\end{aligned}$$

FOCs

Differentiation of time t Lagrangian yields:

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial C_{it}(s^t)} : \quad & U_i'(C_{it}(s^t)) = \lambda_{it}(s^t) \\ \frac{\partial \mathcal{L}_t}{\partial A_{it}(s^{t+1})} : \quad & \lambda_{it}(s^t)Q_t(s^{t+1}) = \mathbb{E}_t[\beta \lambda_{it+1}(s^{t+1})I(s^{t+1})].\end{aligned}$$

where $I(s^{t+1})$ is an indicator for whether state s^{t+1} occurs.

The latter of these can be rewritten:

$$\lambda_{it}Q_t(s^{t+1}) = \beta p_t(s^{t+1})\lambda_{it+1}(s^{t+1}).$$

where $p_t(s^{t+1})$ is the time t probability of state s^{t+1} occurring. □

See Sims Lecture Notes for more general cookbook.

Euler Equation

Combining equations to eliminate λ_{it} we get

$$Q_t(s^{t+1})U_i'(C_{it}(s^t)) = \beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1})).$$

This is a version of the **consumption Euler Equation**.

Trades off consumption today and consumption in one particular state tomorrow.

Cost today: $Q_t(s^{t+1})U_i'(C_{it}(s^t))$.

Expected benefit tomorrow: $\beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))$.

Since Euler equation holds for each state S^{t+1}, it also holds on average.

$$
\begin{aligned}
\sum_{s^{t+1}|s^t}[Q_t(s^{t+1})U_i'(C_{it}(s^t))] &= \sum_{s^{t+1}|s^t}[\beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))] \\
\Rightarrow U_i'(C_{it}(s^t))\sum_{s^{t+1}|s^t}Q_t(s^{t+1}) &= \beta\mathbb{E}_t[U_i'(C_{it+1}(s^{t+1}))] \\
\Rightarrow U_i'(C_{it}(s^t))(1+R_{ft}(s^t))^{-1} &= \beta\mathbb{E}_t[U_i'(C_{it+1}(s^{t+1}))] \\
\Rightarrow U_i'(C_{it}(s^t)) &= \beta(1+R_{ft})\mathbb{E}_t[U_i'(C_{it+1}(s^{t+1}))]
\end{aligned}
$$

where $R_{ft}(s^t)$ is the riskless interest rate in state s^t.

Notice that buying one unit of each Arrow security is the same as buying a riskless bond.

3.3.4 Transversality Condition

In finite horizon case, there was a complementary slackless condition that said that household should not die with positive wealth.(i.e.,$u'(c_t)A_t \leq 0 \Rightarrow A_t = 0$.)

Transversality condition:

$$\lim_{j=0}\beta^j\mathbb{E}_t[U_i'(C_{it+j}(s^{t+j}))A_{it+j}(s^{t+j})] \leq 0.$$

Intuitively:

Cannot be optimal to choose a plan that leaves resources with positive net present value today unspent in the infinite future.

Cannot be optimal to allow your wealth to explode at a rate faster than discounted marginal utility is falling.

Transversality and **No Ponzi** are VERY DIFFERENT in future.

No Ponzi: Debt cannot explode. Constraint imposed by lenders.

Transversality: Wealth cannot explode(too fast). Necessary condition for optimality.

Risk Sharing

Complete markets and common beliefs imply perfect risk sharing.

The consumption Euler equation

$$Q_t(s^{t+1})U_t'(C_{it}(s^t)) = \beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1})).$$

holds for all households.

This implies

$$\frac{Q_t(s^{t+1})}{\beta p_t(s^{t+1})} = \frac{U_i'(C_{it+1}(s^{t+1}))}{U_i'(C_{it}(s^t))} = \frac{U_k'(C_{kt+1}(s^{t+1}))}{U_k'(C_{kt}(s^t))}.$$

Taking the ratio of this equation for states s^{t+1} and s^{*t+1} yields

$$\frac{U_i'(C_{it+1}(s^{t+1}))}{U_i'(C_{it+1}(s^{*t+1}))} = \frac{U_k'(C_{kt+1}(s^{t+1}))}{U_k'(C_{kt+1}(s^{*t+1}))}.$$

Ratio of marginal utility of all households perfectly correlated. This is called **perfect risk sharing**.

4 Neoclassical Growth Model

Neoclassical Growth Model is also called **Ramsey Model** or **Cass-Koopmans Model**.

Solow model: constant saving rate.

Ramsey or Cass-Koopmans Model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings. This model specifies the preference orderings of individuals and derives their decisions from these preference. It also

Enables better understanding of the factors that affect savings decisions.

Enables to discuss the **optimality** of equilibria.

Clarifies whether the (competitive) equilibria of growth models can be **improved upon**.

Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

4.1 Preliminaries

Consider an economy consisting of a unit measure of infinitely-lived households.(i.e.,an uncountable number of households:e.g.,the set of households H could be represented by the unit interval$[0, 1]$.)

Emphasize that each household is infinitesimal and will have no effect on aggregates.

Can alternatively think of H as a countable set of the form $H\{1, 2, \cdots, M\}$ with $M = \infty$, without any loss of generality.

Advantage of unit measure: averages and aggregates are the same.

4.1.1 Time Separable Preferences

Standard assumptions on preference orderings so that they can be represented by utility functions.

In addition, time separable preferences> each household i has an instantaneous (Bernoulli) utility function (or felicity function)

$$u_i(c_i, t).$$

$u_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave and $c_{i,t}$ is the consumption of household i.

4.1.2 Infinite Horizon and the Representative Household

Given by the von Neumann-Morgenstern expected utility function

$$\mathbb{E}_0^i \sum_{t=0}^{T} \beta_i^t u_i(c_i, t).$$

where $\beta_i \in (0, 1)$ is the discount factor of household i, where $T < \infty$ or $T = \infty$, corresponding to finite planning horizon (e.g., with overlapping generations) or infinite planning horizon.

Exponential discounting and time separability also ensure **time-consistent** behavior.

A solution $\{x_t\}_{t=0}^T$ (possibly with $T = \infty$) is time consistent if: whenever $\{x_t\}_{t=0}^T$ is an optimal solution starting at time $t = 0$, $\{x_t\}_{t=t'}^T$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$.

4.1.3 Challenges to the Representative Household

An economy admits a representative household if preference side can be represented as if a single household made the aggregate consumption and saving decisions subject to a single budget constraint.

Simplest case that will lead to the existence of a representative household: suppose each household is identical.

But we can do a little, but not much, better than that: aggregation problems and Gorman preferences (out of the scope of this course, but some intuition here).

4.2 Preference, Technology and Demographics

Infinite-horizon, continuous time.

Representative household with instantaneous(瞬时) utility function

$$u(c_t).$$

Assumption $u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'' , and satisfies the following Inada type assumptions

$$\lim_{c\to 0} u'(c) = \infty; \lim_{c\to\infty} u'(c) = 0.$$

Suppose representative household represents set of identical households(normalized to 1).

Each household has an instantaneous utility function given by the above utility function.

$L_0 = 1$ and

$$L_t = \exp(nt).$$

All members of household supply their labor inelastically.

Objective function of each household at $t = 0$

$$U_0 \equiv \int_0^\infty e^{-\rho t} L_t u(c_t) \mathrm{d}t = \int_0^\infty e^{-(\rho-n)t} u(c_t) \mathrm{d}t.$$

where c_t consumption per capita at t, ρ is the subjective discount rate, and effective discount rate is $\rho - n$.

Continues time analogue of $\sum_{t=0}^{\infty} \beta_i^t u_i(c_i, t)$.

Objective function embeds:

(a) Household is fully altruistic towards all of its future members, and makes allocations of consumption(among household members) cooperatively.

(b) Strictly concave of $u(\cdot)$.

Thus each household member will have an equal consumption

$$c_t \triangleq \frac{C_t}{L_t}.$$

Utility of $u(c_t)$ per household member at time t, total of $L_t u(c_t) = \exp(nt)u(c_t)$.

With discount at rate of $\exp(-\rho t)$, obtain U_0.

Assumption $\rho > n$.

Ensures that in the model without growth, discounted utility is finite(otherwise infinite utility are not well behaved equilibrium). Will strengthen it in model with growth.

Start model without any technological progress.

Factor and product markets are competitive.

Production possibilities aet of the economy is represented by

$$Y_t = F(K_t, L_t).$$

Standard constant returns to scale and **Inada assumptions** still hold.

Per capita production function

$$y_t \triangleq \frac{Y_t}{L_t} = F\left[\frac{K_t}{L_t}, 1\right] = f(k_t).$$

where, as before, k_t is capital per capita.

Competitive factor markets then imply

$$\begin{aligned} R_t &= F_K[K_t, L_t] = f'(k_t) \\ w_t &= F_L[K_t, L_t] = f(k_t) - k_t f'(k_t). \end{aligned}$$

Denote asset holdings of the representative household at time t by $\mathcal{A}_t$, then

$$\dot{\mathcal{A}}_t = r_t \mathcal{A}_t + w_t L_t - c_t L_t.$$

r_t is the risk-free market flow rate of return on assets, and $w_t L_t$ is the flow of labor income earnings of the household.

Defining per capita assets as

$$a_t \triangleq \frac{\mathcal{A}_t}{L_t}.$$

We obtain

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t.$$

Household assets can consist of capital stock, K_t which they rent to firms and government bonds B_t.

With uncertainty, households would have a portfolio(投资组合) choice between K_t and riskless bonds.

With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.

(The total net supply of bonds is zero in a closed economy without government) Thus, market clearing

$$a_t = k_t.$$

No uncertainty depreciation rate of δ implies

$$r_t = R_r - \delta.$$

4.3 The Budget Constraint

The differential equation

$$\dot{a_t} = (r_t - n)a_t + w_t - c_t.$$

is a lower bound on assets.

Three options:

(1) Lower bound on assets such as $a_t \geq 0$ for all t.

(2) Natural debt limit.

(3) No Ponzi Game Condition.

4.4 The No Ponzi Game Condition

Infinite-horizon no Ponzi game condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)\mathrm{d}s} \geq 0.$$

Transversality condition ensures individual would never want to have positive wealth asymptotically, so no Ponzi game condition can be strengthened to (though not necessary in general)

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)\mathrm{d}s} = 0.$$

4.4.1 *Understanding the No Ponzi Game Condition

Write the single budget constraint of the form

$$\int_0^T c_t L_t \exp\left(\int_t^T \mathrm{d}s\right) \mathrm{d}t + A_T = \int_0^T w_t L_t \exp\left(\int_t^T \mathrm{d}s\right) \mathrm{d}t + A_0 \exp\left(\int_t^T \mathrm{d}s\right).$$

Differentiating with respect to T and dividing L_t gives the asset evolving equation (budget constraint).

Now imagine that the above equation applies to a finite-horizon economy. Flow budget constraint (asset) does not guarantee that $A_T \geq 0$. Thus in finite-horizon we would simply impose the above equation as a boundary condition. The no Ponzi game condition is the infinite horizon equivalent of this (obtained by dividing by Lt and multiplying both sides by $\exp\left(\int_t^T rs\mathrm{d}s\right)$ and taking the limit as $T \to \infty$).

4.4.2 *Definition of Equilibrium

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c_t, k_t, w_t, R_t]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock K_0,and the time path of prices $[w_t, R_t]_{t=0}^{\infty}$, and all markets clear.

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c_t, k_t, w_t, R_t]_{t=0}^{\infty}$, such that the representative household maximizes $U_0 = \displaystyle\int_0^{\infty} e^{-(\rho-n)t} u(c_t)\mathrm{d}t$, subject to $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ and $\lim\limits_{t\to\infty} a_t e^{\int_0^t (r_s - n)\mathrm{d}s} \geq 0$ given initial capital-labor ratio k_0. Factor prices are given by $R_t = F_K[K_t, L_t] = f'(k_t)$ and $w_t = F_L[K_t, L_t] = f(k_t) - k_t f'(k_t)$. and the rate of return on assets given by $r_t = R_t - \delta$.

4.5 Household Maximization

set up the current-value Hamitonian

$$H(a, c, \mu) = u(c_t) + \mu_t[(r_t - n)a_t + w_t - c_t].$$

Maximum Principle $\to$ **candidate solution**.

$$\begin{aligned}
H_c(a, c, \mu) &= u'(c_t) - \mu_t = 0 \\
H_a(a, c, \mu) &= \mu_t(r_t - n) = -\dot{\mu}_t + (\rho - n)\mu_t \\
\lim_{t\to\infty}[e^{-(\rho-n)t}\mu_t a_t] &= 0 \\
\dot{a}_t &= (r_t - n)a_t + w_t - c_t
\end{aligned}$$

The second equation is **Continuous-Time Euler equation**. The third equation is a combination of **No Ponzi and transversality condition**, and the last equation is **budget constraint**. The last two equations' limits are formed at both ends and flow.

For any $\mu_t > 0$, $H_c(a, c, \mu)$ is a concave function of (a, c) and strictly concave in c.

The first necessary condition implies $\mu_t > 0$ for all t.

Therefore, sufficient conditions imply that the candidate solution is an optimum.

Rearrange the second condition

$$\frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho).$$

FOC implies that

$$u'(c_t) = \mu_t.$$

Differentiate with respect to time and divide by μ_t

$$u''(c_t)\dot{c}_t = \dot{\mu}_t \Rightarrow \frac{c_t \cdot u''(c_t)\dot{c}_t}{u'(c_t) = \mu_t \cdot c_t} = \frac{\dot{\mu}_t}{\mu_t} \Rightarrow \frac{c_t \cdot u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t}.$$

Conbine the last two equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(r_t - \rho).$$

where

$$-\frac{c_t \cdot u''(c_t)}{u'(c_t)} \triangleq \varepsilon(c_t).$$

is the elasticity of the marginal utility $u'(c_t)$.

Consumption will grow over time when the discount rate is less than the rate of return an assets.

Speed at which consumption will grow is related to the elasticity of marginal utility of consumption $\varepsilon(c_t)$.

Even more importantly, $\varepsilon(c_t)$ is the inverse of the intertemporal elasticity of substitution(跨期替代弹性).

Integrating $\frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho)$,

$$\begin{aligned}
\int_0^t \frac{\dot{\mu}_t}{\mu_t}\mathrm{d}t &= -\int_0^t (r_s - \rho)\mathrm{d}s \\
\Rightarrow \ln \mu_t\big|_0^t &= -\int_0^t (r_s - \rho)\mathrm{d}s \\
\Rightarrow \mu_t &= \mu_0 e^{-\int_0^t (r_s - \rho)\mathrm{d}s} = u'(c_0) e^{-\int_0^t (r_s - \rho)\mathrm{d}s}.
\end{aligned}$$

Substituting into transversality condition

$$0 = \lim_{t\to\infty}\left[e^{-(\rho - n)t} a_t u'(c_0) e^{-\int_0^t (r_s - \rho)\mathrm{d}s}\right].$$

if and only if

$$0 = \lim_{t\to\infty}\left[a_t e^{-\int_0^t (r_s - n)\mathrm{d}s}\right].$$

This is the **strong version** of the no-Ponzi Condition.

Since $a_t = k_t$, transversality condition is also equivalent to

$$\lim_{t\to\infty}\left[k_t e^{-\int_0^t (r_s - n)\mathrm{d}s}\right] = 0.$$

Notice term $e^{-\int_0^t (r_s - n)\mathrm{d}s}$ is a present-value factor: converts a unit of income at t to a unit of income at 0;

When $r_s = r$, factor would be e^{-rt}. More generally, define an average interest rate between dates 0 and t.

$$\bar{r_t} = \frac{1}{t}\int_0^t r_s \mathrm{d}s.$$

Thus conversion factor between dates 0 and t is $e^{-\bar{r_t}t}$. And the transversality condition

$$\lim_{t\to\infty}\left[e^{-(\bar{r_t} - n)t} a_t\right] = 0.$$

4.6 Equilibrium Prices

Equilibrium prices given by equations about factor prices. Thus market rate of return for consumers

$$r_t = f'(k_t) - \delta.$$

Substituting this into the consumer's problem, we have

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(f'(k_t) - \delta - \rho).$$

4.7 Optimal Growth

In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility(可行性) constraints

$$\max_{[k_t, c_t]_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} u(c_t) \mathrm{d}t.$$

subject to

$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t.$$

and $k_0 > 0$.

Again set up the current-value Hamiltonian

$$H(k, c, \mu) = u(c_t) + \mu_c[f(k_t) - (n+\delta)k_t - c_t].$$

Candidate solution from the Maximum Principle

$$\begin{aligned} H_c(k, c, \mu) &= u'(c_t) - \mu_t = 0 \\ H_k(k, c, \mu) &= \mu_t(f'(k_t) - \delta - n) = -\dot{\mu}_t + (\rho - n)\mu_t \\ \lim_{t \to \infty} \left[e^{-(\rho-n)t\mu_t k_t} \right] &= 0. \end{aligned}$$

Hamiltonian function is strictly concave on k. From **Sufficient Theorem**, this is the only solution. Repeating the same steps as before, these imple

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(f'(k_t) - \delta - \rho).$$

And the transversality condition

$$\lim_{t \to \infty} \left[k_t e^{-\int_0^t (f'(k_s) - \delta - n) \mathrm{d}s} \right] = 0.$$

4.8 Steady State Equilibrium

Steady State Equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus

$$\dot{k}_t = 0 \text{ and } \dot{c}_t = 0.$$

As long as $f(k^*) > 0$, irrespective of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta.$$

Pins down the steady state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.

Theorem **[Modified Golden Rule]:** Level of the capital stock that does not maximize steady state consumption, because earlier consumption is preferred to later consumption.

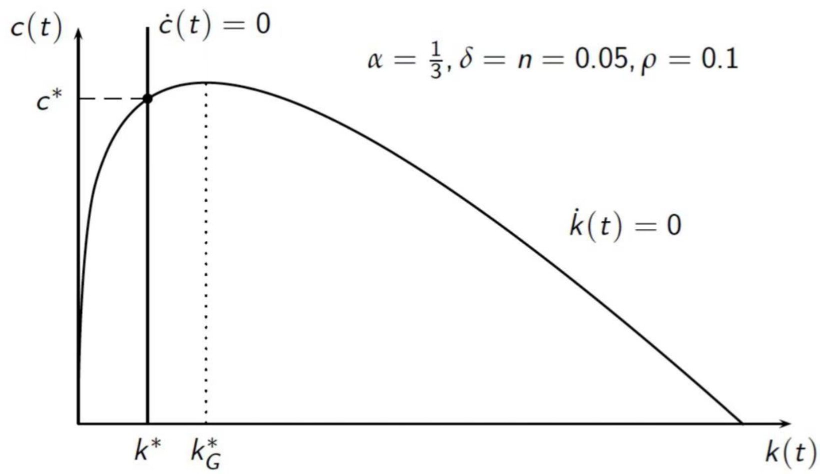

Figure 8 Steady state in the baseline neoclassical growth model

Given k^*, steady state consumption level

$$c^* = f(k^*) - (n + \delta)k^*.$$

A steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.

Loosely, lower discount rate implies greater patience and thus greater savings.

Without technological progress, the steady state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}.$$

Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model. Result depends on the way which intertemporal discounting takes place.

k^* and thus c^* do not depend on the instantaneous utility function $u(\cdot)$. Form of the utility function only affects the transitional dynamics. Not true when there is technological change.

4.9 Transitional Dynamics

Equilibrium is determined by two differential equations

$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t$$
$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(f'(k_t) - \delta - \rho)$$

Moreover, we have an initial condition k_0, also a boundary conditional at infinity

$$\lim_{t\to\infty}\left[k_t e^{-\int_0^t (f'(k_s)-\delta-n)\mathrm{d}s}\right] = 0.$$

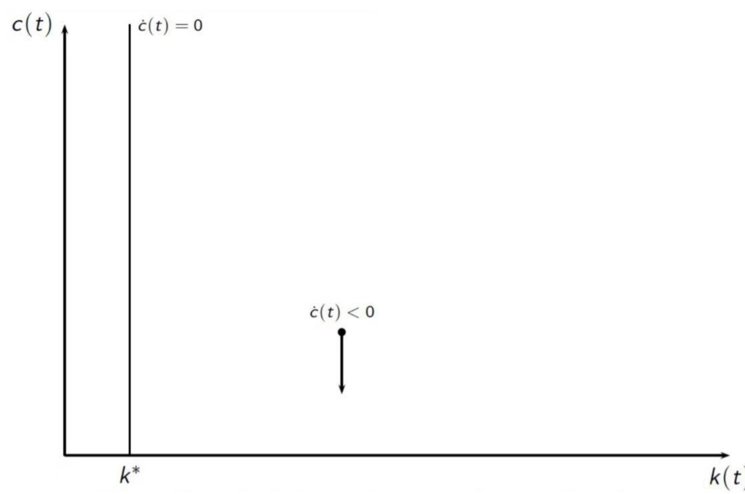

Figure 9 Dynamics in the baseline neoclassical growth model

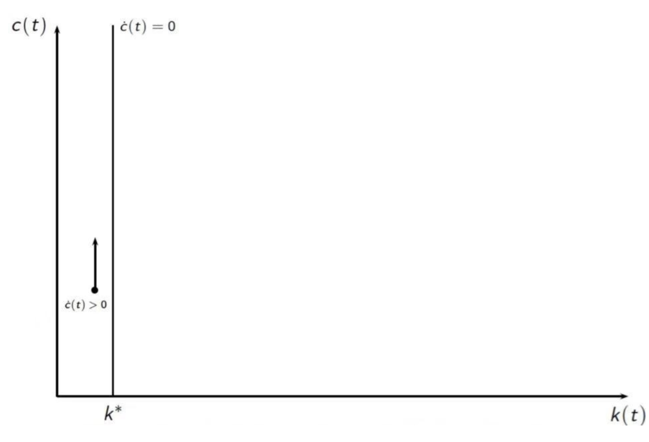

Figure 10 Dynamics in the baseline neoclassical growth model

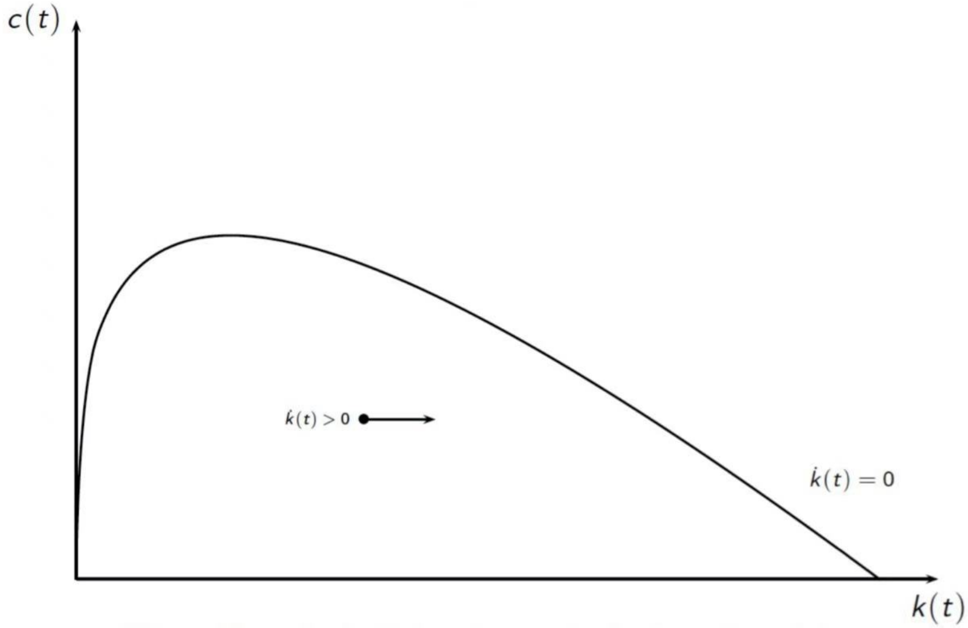

Figure 11 Dynamics in the baseline neoclassical growth model

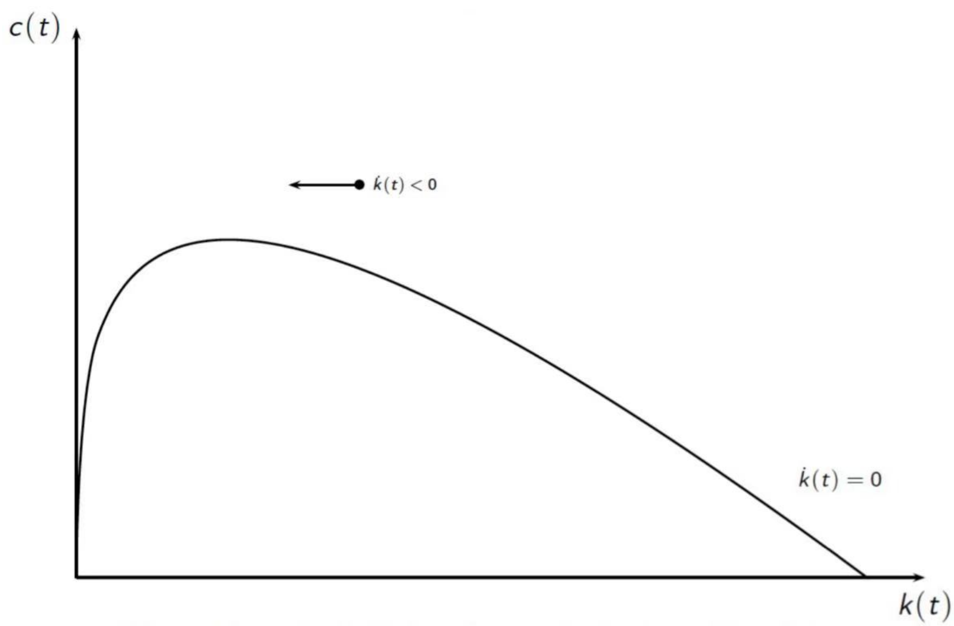

Figure 12 Dynamics in the baseline neoclassical growth model

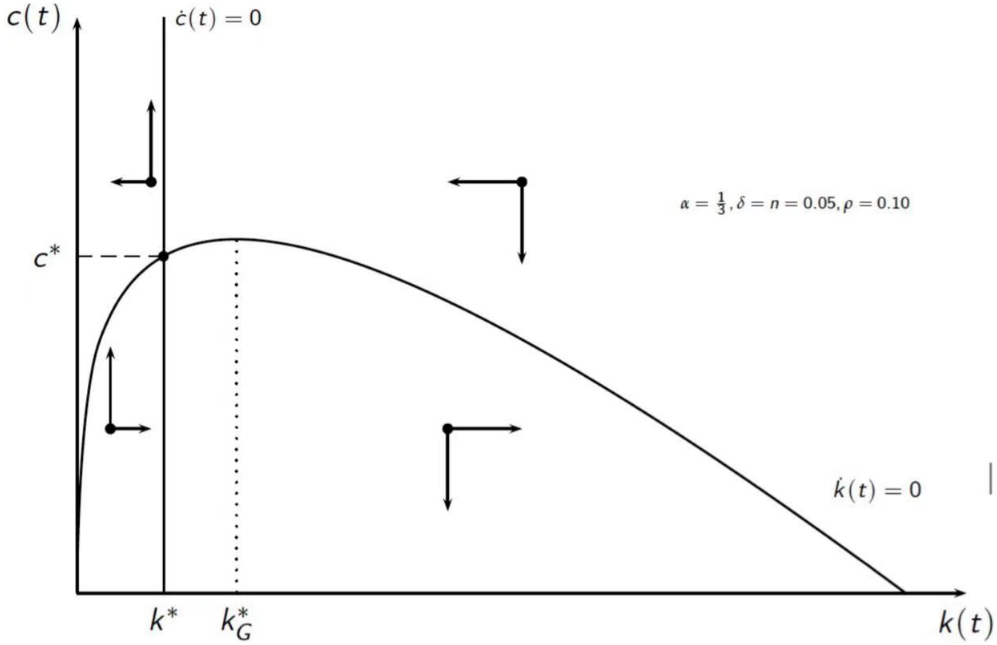

Figure 13 Dynamics in the baseline neoclassical growth model

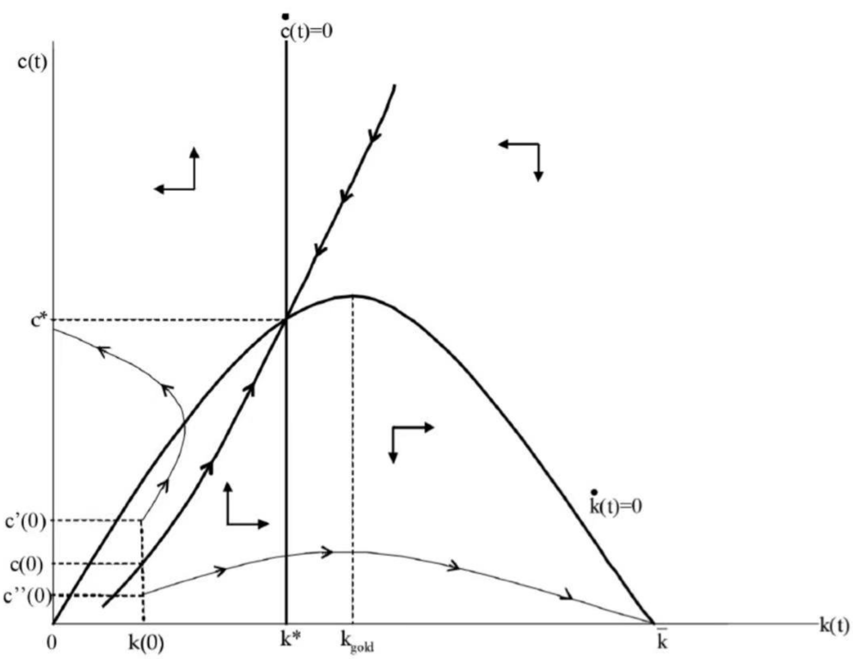

Figure 14 Transitional dynamics in the baseline neoclassical growth model

4.10 Technological Change

Extend the production function to

$$Y_t = F[K_t, A_t L_t].$$

where

$$A_T = \exp(gt) A_0.$$

A consequence of Uzawa Theorem: This equation imposes purely labor-augmenting-Harrod-neutral-technological change.

Define

$$\hat{y}_t \triangleq \frac{Y_t}{A_t L_t} = f\left[\frac{K_t}{A_t L_t}, 1\right] \triangleq f(k_t).$$

where

$$k_t \triangleq \frac{K_t}{A_t L_t}.$$

Also need to impose a further assumption on preferences in order to ensure balanced growth.

Define balanced growth as a pattern of growth consistent with the Kaldor facts of constant capital-output ratio and capital share in national income.

These two observations together also imply that the rental rate of return on capital, R_t has to be constant, which from $r_t = R_t - \delta$ implies that r_t has to be constant.

Again refer to an equilibrium path that satisfies these conditions as a balanced growth path(BGP).

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(r_t - \rho).$$

If $r_t \to r^*$, then $\dfrac{\dot{c}_t}{c_t} \to g_c$ is only possible if $\varepsilon(c_t) \to \varepsilon$. (The elasticity of marginal utility of consumption is asymptotically constant.)

Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with

$$u(c) = \begin{cases} \dfrac{1}{1-\theta} c_t^{1-\theta}, & \theta > 0 \text{ and } \theta \neq 0 \\ \ln c_t, & \theta = 1 \end{cases}$$

Elasticity of marginal utility of consumption ε is given by θ.

When $\theta = 0$, these represent linear preferences. When $\theta = 1$, we have log preferences, and as $\theta \to \infty$ infinitely risk-averse, and infinitely unwilling to substitute consumption over time.

Assume that the economy admits a representative household with CRRA preferences

$$\int_0^{\infty} e^{-(\rho - n)t} \frac{\tilde{c}_t^{\,1-\theta} - 1}{1-\theta} \mathrm{d}t.$$

$\tilde{c}_t \triangleq \dfrac{C_t}{L_t}$ is per capita consumption.

Refer to this model, with labor-augmenting technological change and CRRA preference as given by this equation as the canonical model.

Euler equation takes the simpler form

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(r_t - \rho).$$

Steady state equilibrium first: since with technological progress there will growth in per capita income, $\tilde{c}_t$ will grow.

Instead define

$$c_t \triangleq \frac{C_t}{A_t L_t} \triangleq \frac{\tilde{c}_t}{A_t}.$$

This normalized consumption level will remain constant along the BGP

$$\frac{\dot{c}_t}{c_t} \triangleq \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} - g = \frac{1}{\theta}(r_t - \rho - \theta g)k_t.$$

where $k_t \triangleq \dfrac{K_t}{A_t L_t}$.

Transversality condition in turn, can be expressed as

$$\lim_{k\to\infty} \left\{ k_t e^{-\int_0^t \left[f'(k_s) - g - \delta - n\right] \mathrm{d}s} \right\} = 0.$$

In addition, equilibrium r_t is similarly determined as before, so

$$r_t = f'(k_t) - \delta.$$

Since in steady state c_t must remain constant

$$r_t = \rho + \theta g.$$

or

$$f'(k^*) = \rho + \delta + \theta g.$$

Pins down the steady state value of the normalized capital ratio k^*

$$c^* = f(k^*) - (n + g + \delta)k^*.$$

Per capita consumption grows at the rate g.

Because there is growth, to make sure that the transversality condition is in fact satisfied substitute.

Combine the last two equation, we have

$$\lim_{t\to\infty} \left\{ k_t e^{-\int_0^t [\rho - (1-\theta)g - n] \mathrm{d}s} \right\} = 0.$$

can only if $\rho - (1-\theta)g - n > 0$, or alternatively

$$\rho - n > (1-\theta)g.$$

Recall in steady state $r = \rho + \theta g$ and the growth rate of output is $g + n$. Therefore, equivalent to requiring that $r > g + n$.

Steady state capital-labor ratio no longer independent of preferences, depends on θ.

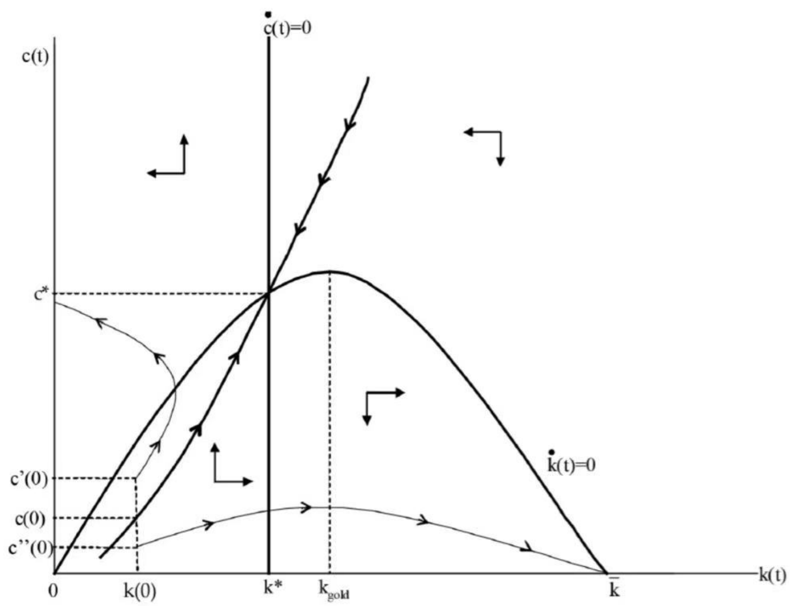

Figure 15 Transitional dynamics in the neoclassical growth model with technological change

4.11 Comparative Dynamics

Comparative statics: change in steady state in response to changes in parameters.

Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.

Look at the effect of a change in tax on capital(or discount rate ρ).

Consider the neoclassical growth in steady state (k^*, c^*). Tax declines $\tau' < \tau$.

Let the new steady state equilibrium be (k^{**}, c^{**}). Since $\tau' < \tau$, $k^{**} > k^*$ while the equilibrium growth rate will remain unchanged.

4.12 The Role of Policy

Example of effect of differences in polices.

Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers.

Capital accumulation equation remains as above

$$\dot{k}_t = f(k_t) - c_t - (n + g + \delta)k_t.$$

But interest rate faced by households changes to

$$r_t = (1 - \tau)(f'(k_t) - \delta).$$

Growth rate of normalized consumption is then obtained from the consumer Euler equation

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}(r_t - \rho - \theta g) \\ &= \frac{1}{\theta}((1 - \tau)(f'(k_t) - \delta) - \rho - \theta g)\end{aligned}$$

Identical argument to that before implies

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau}.$$

Higher τ, since $f'(\cdot)$ is decreasing, reduces k^*.

Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.

But have not so far offered a reason why some countries may tax capital at a higher rate than others.

4.13 Conclusions

Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.

Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.

Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.

Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.

This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.

But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.

5 Basics for Dynamic Stochastic General-Equilibrium Model

5.1 DSGE

Dynamic: The effect of current choices on future uncertainty makes the models dynamic and assigns a certain relevance to the expectations of agents in forming macroeconomic outcomes.

Stochastic: The models take into consideration the transmission of random shocks into the economy and the subsequent economic fluctuations.

General: Referring to the entire economy as a whole(within the model) in that price levels and output levels are determined jointly. As opposed to a Partial equilibrium where price-levels are taken as given and only output-levels are determined within the model economy.

Equilibrium: Subscribing to the Walresian, General Competitive Equilibrium Theory, the model captures the interaction between policy actions and subsequent behavior of agents'.

5.2 The Deterministic Growth Model

Our interest is in the problems of the form

$$
\begin{aligned}
V^*(x_0) = &\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
\text{s.t.} \quad & x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \cdots \\
& x_0 \in X \text{ is given.}
\end{aligned}
$$

The mapping $\Gamma : X \to Y$ is a correspondence: for $\forall x \in X$ it assigns a set $\Gamma(x) \subset Y$.

$$
\begin{aligned}
V^*(x_0) = &\max_{k_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & k_{t+1} = (1-\delta)k_t + i_t \\
& y_t = f(k_t) \\
& y_t = c_t + i_t \\
& c_t, k_{t+1} \geq 0, \; k_0 \text{ is given.}
\end{aligned}
$$

Sequential form and Bellman(recursive) form.

With full depreciation assumption $\delta = 1$:

$$
\begin{aligned}
V^*(x_0) = &\max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
\text{s.t.} \quad & k_{t+1} \in \Gamma(k_t), \quad t = 0, 1, 2, \cdots \\
& k_0 \in X \text{ is given.}
\end{aligned}
$$

Bellman's Principle of Optimality implies we can write this:

$$V(k) = \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta V(k')\}, \ \forall k \in X, \ k_0 \text{ is given.}$$

where $k' \triangleq k_{t+1}$.

5.3 Stochastic Dynamic Programming

Our goal: to set-up and solve a problem like this:

$$V(k, z) = \max_{k' \in \Gamma(k,z)} \{F(k, k', z) + \beta \mathbb{E}[V(k', z')|z]\}.$$

z_t is a stochastic component.

We need to specify some stochastic process for z_t.

5.3.1 Markov Chains

Let's consider cases where the stochastic component can take finitely many values(discrete-state process).

x_t will be a Markov chain

$$x \in S = \{x_1, x_2, \cdots, x_n\}.$$

x_i refers to the realization of an event.

This means it has the Markov property

$$P(x_{t+1} \in S|x_t, x_{t+1}, \cdots, x_{t-k}) = P(x_{t+1}|x_t).$$

Future values depend only on the current value. Useful for using recursive techniques.

We'll consider time-invariant chains: fixed probabilities of moving from one state to another.

The stochastic process x_t will be a sequence of random vectors.

The n dimensional state space consists of vector $e_i, i = 1, \cdots, n$.

$n \times 1$ unit vector that records the position of the system. e.g.

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$n \times n$ transition/stochastic matrix $\mathbf{P}$ records the probabilities of moving from one state to another in one period.

$$\mathbf{P}_{ij} = P(x_{t+1} = e_j|x_t = e_i).$$

To be probabilities, for $i = 1, \cdots, n$, the matrix must satisfy

$$\sum_{j=1}^{n} \mathbf{P}_{ij} = 1$$
$$\mathbf{P}_{ij} \geq 0$$

There needs to be an initial probability distribution π_0.

An Example

How does Markov chain x_t relate to the state variable we care about, e.g. TFP z_t? How would you forecast it?

Suppose GDP growth, y_t can be in boom or bust. The boom state implies $y_t = 1.2$ and the recession state $y_t = -0.4$. x_1 indicates we are in a boom, x_2 in a recession.

e.g. Hamilton(1989)

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$$

5.3.2 Probability Distribution over Time

Define π_t^T as the unconditional probability distribution of x_t, $(1 \times n)$ whose i^{th} element is $P(x_t = e_i)$.

From an initial distribution $\pi_0^T = (0, \cdots, 1, \cdots, 0)$, the probability distribution of x_1 is

$$\pi_1^T = P(x_1) = \pi_0^T \mathbf{P}.$$

And for x_2 it is

$$\pi_2^T = P(x_2) = \pi_1^T \mathbf{P} = (\pi_0^T \mathbf{P})\mathbf{P} = \pi_0^T \mathbf{P}^2.$$

In general

$$\pi_k^T = \pi_0^T \mathbf{P}^k.$$

5.3.3 Stationary Distributions

Is there a stationary(or invariant)(固定的) distribution π, that

$$\pi^T = \pi^T \mathbf{P}.$$

If we start with a distribution over states π, tomorrow we end up with the same distribution over states.

There is always at least one stationary distribution. It is an eigenvector(特征向量) associated with the unit eigenvalue of $\mathbf{P}^T$.

$$\pi^T (I - \mathbf{P}) = 0$$
$$(I - \mathbf{P}^T)\pi = 0$$

Markov chain $(\pi, \mathbf{P})$ is stationary, if for a given initial distribution $\pi^T = \pi^T \mathbf{P}$.

5.3.4 Asymptotic Stationarity

Given a π_0, does π_t approach a stationary distribution over time?
i.e. $\lim_{t \to \infty} \pi_0 \mathbf{P}^t = \pi_\infty$ where π_∞ solves

$$(I - \mathbf{P}^T)\pi = 0.$$

If $\forall \pi_0$, $\lim_{t \to \infty} \pi_0 \mathbf{P}^t = \pi_\infty$, we say the Markov chain is **asymptotically stationary** with a unique invariant distribution.

Will be true, from every state there is a positive probability of a=moving to another state in one or more steps.

5.3.5 Markov Processes and Markov Chains

The stochastic process could have a continuous state space.
We'll see some like this before

$$z_{t+1} = \rho z_t + \varepsilon_t.$$

If ε is i.i.d.(独立同分布), then z_t follows a Markov process.
The conditional expectation depends only on the last realization of the process.
Computationally it is useful to discretize continuous state Markov processes as a Markov chain.

5.3.6 Approximation of A Continuous State Markov Process

Choose some extreme values for the process. e.g. r standard deviations from the mean to set the bounds.

Discretize the state space into $z = [z_1, \cdots, z_n]$. The distance between each is δ.
For any two grid points(不动点)

$$\begin{aligned}
\mathbf{P}_{jk} &= P(z_k - \frac{\delta}{2} < \rho z_j + \varepsilon_t < z_k + \frac{\delta}{2}) \\
&= P(z_k - \frac{\delta}{2} - \rho z_j < +\varepsilon_t < z_k - \rho z_j + \frac{\delta}{2}) \\
\mathbf{P}_{jk} &= F\left(\frac{z_k - \rho z_j + \delta/2}{\sigma}\right) - F\left(\frac{z_k - \rho z_j - \delta/2}{\sigma}\right)
\end{aligned}$$

Simple example with i.i.d. shocks

$$P(z_t = z^h | z_{t-1} = z^h) = P(z_t = z^h | z_{t-1} = z^l) = 0.5$$
$$P(z_t = z^l | z_{t-1} = z^h) = P(z_t = z^l | z_{t-1} = z^l) = 0.5$$
$$z^h > z^l$$

If we expand the expectation, what does the Bellman equation look like?

$$V(k,z) = \max_{k' \in \Gamma(k,z_i)} \{F(k,k',z) + \beta \sum_{j=1}^{n} \mathbf{P}_{ij} V(k',z')\}$$

$$V(k,z^h) = \max\{u^h + \beta[\mathbf{P}_{hh} V(k',z^h) + \mathbf{P}_{hl} V(k',z^l)]\}$$

$$V(k,z^l) = \max\{u^l + \beta[\mathbf{P}_{lh} V(k',z^h) + \mathbf{P}_{ll} V(k',z^l)]\}$$

5.3.7 Stochastic Dynamic Programming

$$V(k,z_i) = \max_{k' \in \Gamma(k,z_i)} \{F(k,k',z_i) + \beta \sum_{j=1}^{n} \mathbf{P}_{ij} V(k',z_j)\}$$

In general, all the proofs you saw for the deterministic case can be applied to the stochastic case.

More generally,

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \{F(k,k',z) + \beta \mathbb{E}(V(k',z')|z)\}$$

z could be e.g., a finite Markov chain or an $AR(1)$ process.(The latter continuous case requires added steps to the proofs, but think of a discrete approximation.)

An example

$$V(k,z) = \max_{k'} \{u(zk^\alpha - k' + (1-\delta)k) + \beta \mathbb{E}(V(k',z')|z)\}, \quad \forall (z,k)$$

z is a bounded, random variable that follows a first order Markov process(e.g. an $AR(1)$ process).

There exists a maximum possible capital stock such that consumption is non-negative. Provided there is discounting $\beta < 1$ and that the stocks follows a bounded first-order Markov process, there exists a unique value function.

We are interested in finding the policy function $k' = \phi^k(k,z)$.

Is there a steady state

If z_t does not have a degenerate distribution, k_t will not converge to a single number $k' = \phi^k(k,z)$, it will converge to an invariant limiting distribution.

At sufficiently fay away horizons, k should be independent of k_0.

The average value in this distribution will be the same as the time average of k_t as $T \to \infty$.

The stochastic process for the capital stock is therefore ergodic(遍历性).

Instead of a steady state, we have an invariant limiting distribution for capital, output, etc.

5.4 Solving the Stochastic Growth Model

Consider the stochastic growth model again:

$$V(k,z) = \max_{k'} \{u(zk^\alpha - k' + (1-\delta)k) + \beta \mathbb{E}(V(k',z')|z)\}, \quad \forall (z,k)$$

We want to solve the model to find:

The value function itself.

The policy functions:

$$c = \phi^c(k, z)$$
$$k' = \phi^k(k, z)$$

We want to solve for the endogenous variables only as functions of the state each period(and the deep parameters).

Solution Methods:

Guess and Verify: only works in limited cases.

*Value function iteration

*Linearization: undetermined coefficients(待定系数法) and eigenvalue decomposition(特征值分解).

5.5 Guess and Verify

If we know what form the solution takes, we can uses this as a guess, find the unknown coefficients and verify it is a solution.

Two options:

Guess and verify the value function, deriving the policy function along the way.

Guess and verify the policy function directly.

Works well, but only for very special cases $u(c) = \ln c$ and $\delta = 1$.

5.5.1 Envelop Theorem in Our Simplest Setting

We assume that $f(k) = k^\alpha$, and depreciation rate $\delta = 1$.

We want to choose k' to maximize

$$U = u(k^\alpha - k') + \beta V(k').$$

FOC:

$$0 = -u'(k^\alpha - k') + \beta \frac{\mathrm{d}V(k')}{\mathrm{d}k'}.$$

We can write the value function as

$$V(k) = \max_{k'} \{u(k^\alpha - k') + \beta V(k')\}.$$

$$\begin{aligned}
\frac{\mathrm{d}V(k)}{\mathrm{d}k} &= \alpha k^{\alpha-1} u'(k^\alpha - k') - u'(k^\alpha - k') \frac{\mathrm{d}k'}{\mathrm{d}k} + \beta \frac{\mathrm{d}V(k')}{\mathrm{d}k'} \frac{\mathrm{d}k'}{\mathrm{d}k} \\
&= \alpha k^{\alpha-1} u'(k^\alpha - k') + \left\{ -u'(k^\alpha - k') + \beta \frac{\mathrm{d}V(k')}{\mathrm{d}k'} \right\} \frac{\mathrm{d}k'}{\mathrm{d}k} \\
&= \alpha k^{\alpha-1} u'(c)
\end{aligned}$$

5.5.2 Guess and Verify the Value Function

Assume log utility and $\delta = 1$. Let's make an (informed!) guess that the policy function for k' takes the form

$$k' = Qzk^{\alpha}.$$

We'll also make use of the stochastic Euler derived[Envelop Theorem Used]

$$\frac{1}{c} = \beta\mathbb{E}\left(\frac{\alpha z'(k')^{\alpha-1}}{c'}\bigg| z\right).$$

and the resource constraint

$$c = zk^{\alpha} - k'.$$

From the resource constraint, the policy function for consumption is

$$c = (1-Q)zk^{\alpha}.$$

$$\begin{aligned}
\frac{1}{(1-Q)zk^{\alpha}} &= \beta\mathbb{E}\left\{\frac{\alpha z'(k')^{\alpha-1}}{(1-Q)z'k'^{\alpha}}\bigg| z\right\} \\
\frac{1}{zk^{\alpha}} &= \beta\alpha(k')^{-1} \\
k' &= \alpha\beta zk^{\alpha}
\end{aligned}$$

Guess a form for the value function

$$V(k,z) = G + B\ln(k) + D\ln(z).$$

Use the FOCs from the Bellman equation and from the guess with respect to k' to find the general form of the policy function.

Substitute this, and our guess into the Bellman equation.

Solve for the unknowns G, B, D.

Bellman Equation

$$V(k,z) = \max_{k'}\{u(zk^{\alpha} - k') + \beta\mathbb{E}(V(k',z')|z)\}.$$

FOC

$$\frac{-1}{zk^{\alpha} - k'} + \beta\frac{B}{k'} = 0 \Rightarrow \beta B(zk^{\alpha} - k') = k'.$$

We have

$$k' = \frac{\beta B}{1+\beta B}zk^{\alpha} = \alpha\beta zk^{\alpha} \Rightarrow B = \frac{\alpha}{1-\alpha\beta}.$$

From

$$V(k,z) = \max_{k'}\{u(zk^{\alpha} - k') + \beta\mathbb{E}(V(k',z')|z)\} = G + B\ln(k) + D\ln(z).$$

And use

$$\ln(z') = \rho_z \ln z + \varepsilon.$$

We have

$$G + D \ln z = \ln(1-\alpha\beta) + \frac{\ln z}{1-\alpha\beta} + \beta G + \frac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta + \beta D \rho_z \ln z.$$

That is

$$\begin{cases} (1-\beta)G = \dfrac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta + \ln(1-\alpha\beta) \\ (1-\beta\rho_z) D \ln z = \dfrac{\ln z}{1-\alpha\beta} \end{cases}$$

So

$$\begin{cases} G = \dfrac{\frac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta + \ln(1-\alpha\beta)}{1-\beta} \\ D = \dfrac{1}{(1-\alpha\beta)(1-\beta\rho_z)} \end{cases}$$

6 Real Business Cycle

6.1 The Basic Neoclassical Model

Preferences

$$\sum_{t=0}^{\infty} b^t [U(C_t, L_t)], \quad b > 0.$$

Where $U_c > 0, U_{cc} < 0; U_l > 0, U_{ll} < 0$. Infinite horizon is an approximation.

Endowments

$$N_t + L_t = 1.$$

Technology

$$Y_t = A_t F(K_t, N_t X_t).$$

of which X_t is the deterministic component of productivity. And

$$X_{t+1} = \gamma X_t, \quad \gamma > 1.$$

The output is used for consumption and investment

$$Y_t = C_t + I_t.$$

The capital stock evolves

$$K_{t+1} = I_t + (1 - \delta) K_t.$$

where δ is the depreciation rate.

Initial conditions

$$K_0 > 0, X_0 > 0, A_0 > 0.$$

Rescale the variables bu X to get rid of the steady state growth. e.g. $y_t \triangleq \frac{Y_t}{X_t}$.

Transformed utility function

$$\sum_{t=0}^{\infty} \beta^t [U(c_t, L_t)].$$

Constraints

$$\begin{aligned} N_t &= 1 - L_t \\ y_t &= A_t F(k_t, N_t) \\ y_t &= c_t + i_t \\ \gamma k_{t+1} &= i_t + (1 - \delta) k_t \end{aligned}$$

Given this close correspondence, RBC analyses sometimes simply start with the transformed economy, omitting growth all together.

6.2 Restrictions

6.2.1 Technology

Production technology

Observations to be matched: constant factor shares and a balanced growth path.

For a balanced growth path to be feasible, technology must be labor-augmenting.

$$Y_t = A_t F(K_t, N_t X_t).$$

Transformed form

$$y_t = A_t k_t^\alpha N_t^{1-\alpha}.$$

TFP process

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t.$$

6.2.2 Preferences

Observations: preferences need to be consistent with long-run growth in macro aggregates, but no trend in hours per worker.

King, Plosser and Rebelo (1988)

$$\begin{aligned}&\frac{1}{1-\sigma}([C_t v(L_t)]^{1-\sigma} - 1), \quad \sigma \neq 1\\&\ln C_t + \ln v(L_t), \quad \sigma = 1\end{aligned}$$

De-trended: $\beta = b(\gamma)^{1-\sigma}$ and $X_0 = 1$

$$\mathbb{E}\sum_{t=0}^{\infty} \beta^t [U(c_t, L_t)].$$

We'll use $v(L) = \exp \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1)$

$$u(c_t, L_t) = \ln c_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1).$$

6.3 Decentralized Equilibrium

6.3.1 Households

$$\mathbb{E}\sum_{t=0}^{\infty} \beta^t [\ln c_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1)].$$

Budget constraint

$$c_t + \gamma k_{t+1} - (1-\delta)k_t + \gamma b_{t+1} = r_t b_t + w_t(1 - L_t) + r_t^k k_t + \Pi_t.$$

P_t is the price of output c_t, for now normalized to 1.
b_{t+1} is holdings of real bond budget at price 1 at time t and yielding r_{t+1}.
r_t is the gross real interest rate between periods $t-1$ and t.
w_t is the real wage, r_t^k is the real rental rate.
Households hold capital.

$$
\begin{aligned}
\mathcal{L} = \mathbb{E} \Bigg\{ & \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t [r_t b_t + w_t(1-L_t) + r_t^k k_t + \Pi_t - c_t - \gamma b_{t+1} \\
& - \gamma k_{t+1} + (1-\delta) k_t] \}
\end{aligned}
$$

$$
\begin{aligned}
\{c_t\} : & \qquad \frac{1}{c_t} = \lambda_t \\
\{L_t\} : & \qquad \theta L_t^{-\eta} - \lambda_t w_t = 0 \\
\{b_{t+1}\} : & \qquad \mathbb{E}_t[-\beta^{t+1}\lambda_{t+1} r_{t+1} + \gamma \beta^t \lambda_t] = 0 \\
& \qquad \mathbb{E}_t \left[\beta \frac{c_t}{c_{t+1}} r_{t+1} \right] = \gamma \\
\{k_{t+1}\} : & \qquad \mathbb{E}_t \left[\beta^{t+1} \lambda_{t+1} \left(r_{t+1}^k + 1 - \delta \right) - \gamma \beta^t \lambda_t \right] = 0 \\
& \qquad \mathbb{E}_t \left[\beta \frac{c_t}{c_{t+1}} \left(r_{t+1}^k + 1 - \delta \right) \right] = \gamma
\end{aligned}
$$

6.3.2 Firms

$$
\max_{N_t, k_t} \{ A_t k_t^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t^k k_t \}.
$$

FOCs

$$
\begin{aligned}
\{N_t\} : & \qquad MPL = w_t = (1-\alpha) \frac{y_t}{N_t} \\
\{k_t\} : & \qquad MPK = r_t^k = \alpha A_t k_t^{\alpha-1} N_t^{1-\alpha}
\end{aligned}
$$

6.4 Terminology(术语)

The **non-stochastic steady state** is defined as a situation in which all variables are constant and where the only source of uncertainty (which is the stochastic part of productivity) is held at its unconditional mean.

A **variable** is a realization of something that can change (either deterministically or stochastically). **Endogenous variables** are variables whose values are determined inside” of a model. **Exogenous variables** are variables whose values are determined outside of a model.

State and control variables: Exogenous variables are always state variable, but endogenous variables can be either controls or states. Loosely, **control variables** are variables whose values are chosen in a model and are free to jump in response to new information.

State variables are variables whose values agents need to know to make decisions. These variables are either exogenous (a productivity term, government spending) or endogenous (capital shocks, stocks of assets, etc).

6.5 The Steady State

We're going to linearize around the deterministic steady state $A = \bar{A} = 1$.

From FOC for capital

$$r^k = \alpha \left(\frac{k}{N}\right)^{\alpha-1} = \frac{\gamma}{\beta} - 1 + \delta$$
$$\Rightarrow \frac{k}{N} = \left(\frac{\alpha}{\frac{\gamma}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}.$$

From the wage equation

$$w = (1-\alpha)\frac{y_t}{N_t} = (1-\alpha)\frac{k_t^\alpha N_t^\alpha}{N_t} = (1-\alpha)\left(\frac{k}{N}\right)^\alpha.$$

From the capital accumulation equation

$$i = (\gamma - 1 + \delta)k.$$

The resource constraint

$$\left(\frac{k}{N}\right)^\alpha N = c + (\gamma - 1 + \delta)k$$
$$\Rightarrow \frac{c}{N} = \left(\frac{k}{N}\right)^\alpha - (\gamma - 1 + \delta)\frac{k}{N}$$

From the intratemporal consumption leisure condition

$$\theta(1-N)^{-\eta} = \frac{1}{c}(1-\alpha)\left(\frac{k}{N}\right)^\alpha$$
$$\frac{c}{N} = \frac{1-\alpha}{\theta}\frac{1-N}{N}\left(\frac{k}{N}\right)^\alpha$$

We consider a special case $\eta = 1$.

Get N

$$N = \frac{\frac{1-\alpha}{\theta}\left(\frac{k}{N}\right)^\alpha}{\frac{1-\alpha+\theta}{\theta}\left(\frac{k}{N}\right)^\alpha - (\gamma - 1 + \delta)\frac{k}{N}}.$$

Steady state investment

$$i = (\gamma - 1 + \delta)\frac{k}{N}N.$$

Steady output

$$y = \left(\frac{k}{N}\right)^{\alpha} N.$$

Steady state consumption

$$c = N\left(\left(\frac{k}{N}\right)^{\alpha} - (\gamma - 1 + \delta)\frac{k}{N}\right).$$

6.6 Linearization

6.6.1 Log Linearization

Taylor Expansion of $f(x)$ around x^*

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \cdots$$

For a sufficiently smooth function, the higher order derivatives will be small.

$$f(x) = f(x^*) + f'(x^*)(x - x^*)$$

Log Linearization: log it first, Taylor expand it then.
Example

$$y_t = A_t k_t^{\alpha} N_t^{1-\alpha}$$

Take logs

$$\ln y_t = \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln N_t$$

Do the Taylor expansion around the steady state values

$$\begin{aligned} \ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln A^* + \frac{1}{A^*}(A_t - A^*) + \\ \alpha \ln k^* + \alpha \frac{1}{k^*}(k_t - k^*) + \\ (1 - \alpha) \ln N^* + (1 - \alpha)\frac{1}{N^*}(N_t - N^*) \end{aligned}$$

With hat indicating percentage deviation from the steady state values, we get

$$\hat{y}_t = \hat{A}_t + \alpha \hat{k}_t + (1 - \alpha)\hat{N}_t$$

If the given form is multiplication, we take the logarithm first and then Taylor's expansion, and if it is addition, we take the differential.

6.6.2 Households

Labor supply

$$-\eta \hat{L}_t = -\hat{c}_t + \hat{w}_t$$

Euler

$$\mathbb{E}_t\left(\hat{c_{t+1}} - \hat{r_{t+1}}\right) = \hat{c}_t$$

Interest rate(gross)

$$r\hat{r}_t = (r - 1 + \delta)\hat{r_t}^k$$

Note: Here we have a addition form, so we choose to take the differential.
From $\hat{x}_t = \dfrac{x_t - x}{x}$, we have

$$x\hat{x}_t = x_t - x = \mathrm{d}x_t.$$

So the

$$r_{t+1}^k + 1 - \delta = r_{t+1}$$

can be

$$\mathrm{d}r_{t+1}^k = \mathrm{d}r_{t+1}.$$

Then

$$r_{t+1}^k \hat{r_{t+1}^k} = r_{t+1}\hat{r_{t+1}} \Rightarrow r_t^k \hat{r_t^k} = r_t \hat{r}_t \Rightarrow r\hat{r}_t = (r - 1 + \delta)\hat{r_t}^k.$$

6.6.3 Firms

Labor demand

$$\hat{w}_t = \hat{A}_t + \alpha\left(\hat{k}_t - \hat{N}_t\right)$$

Capital demand

$$\hat{r_t}^k = \hat{A}_t + (\alpha - 1)\left(\hat{k}_t - \hat{N}_t\right)$$

6.6.4 Technology

Production function

$$\hat{y}_t = \hat{A}_t + \alpha\hat{k}_t + (1 - \alpha)\hat{N}_t$$

Time constraint

$$N\hat{N}_t = -L\hat{L}_t$$

Resource constraint

$$\frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t = \hat{y}_t$$

6.6.5 Law of Motion for the states

Capital

$$\gamma\hat{k_{t+1}} = (1 - \delta)\hat{k}_t + \frac{i}{k}\hat{i}_t$$

Stochastic process for A

$$\hat{A}_t = \rho\hat{A_{t-1}} + \varepsilon_t$$

Summary: 2 states: k, A; 8 controls: c, w, r^k, N, L, y, i, r; 8 equations.

6.6.6 Model Solution

$$\begin{bmatrix} \hat{c}_t \\ \hat{N}_t \\ \hat{L}_t \\ \hat{y}_t \\ \hat{i}_t \\ \hat{w}_t \\ \hat{r_t}^k \\ \hat{r}_t \end{bmatrix} = F \begin{bmatrix} \hat{k}_t \\ \hat{A}_t \end{bmatrix} \qquad \begin{bmatrix} \widehat{k_{t+1}} \\ \widehat{A_{t+1}} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \hat{k}_t \\ \hat{A}_t \end{bmatrix}$$

6.7 Calibrating the Model

The basic idea of calibration is to choose parameter values on the basis of micro-economic evidence and then to compare the model's predictions concerning the variance and covariances of various series with those in the data.

Two advantages of calibration: brings information from micro research; avoid the difficulties in statistical inference and interpretation.

Bad side: no measurement about how good the model is.

6.7.1 Calibrating Model's Parameters

Discount factor $b = 0.984$: steady state real rate coincides with the average return to capital in the economy (S&P 500, 6.5%).

The labor share is around 2/3, $\alpha = 1/3$.

Growth rate of technical change set to match the long-run trends in GDP growth: $\gamma = 1.004$.

Depreciation rate $\delta = 0.025$, set to match the empirical K/Y ratio.

Utility is logarithmic in consumption, this implies an Elasticity of intertemporal substitution of 1.

$\eta = 1, \theta = 3.48$ is chosen to match steady state hours worked, approximately 20% of time available.

The persistence of the A process can be estimated from the de-trended Solow residual in the data $\rho = 0.979$.

Parameters	Description	Value
b	Discount Factor	0.984
θ	Relative importance of leisure	3.48
σ	Risk aversion for consumption	1
η	Risk aversion for leisure	1
γ	Growth rate for labor productivity	1.004
α	Cobb-D production capital share	0.333
δ	Per quarter depreciation rate	0.025
ρ	Persistence of Tech shock	0.979
σ_ϵ	SD of Tech shock	0.0072

6.7.2 Calibrating the Technology Process

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} \left(X_t N_t\right)^{1-\alpha} \\
\ln SR_t &= \ln Y_t - \alpha \ln K_t - (1-\alpha)\ln N_t \\
&= \ln A_t + (1-\alpha)\ln X_t
\end{aligned}
$$

X_t is a deterministic trend. Can de-trend $\ln SR_t$ and then estimate an $AR(1)$.

Success and Criticisms

Technology shocks are the dominant source of fluctuations. And Solow Residual has excessively large variation.

Unreasonable degree of intertemporal substitution in labor supply.

The model's strongly pro-cyclical real wage poses tension with empirical facts. (solved, wage smoothing, etc)

Equity premium is incompatible with standard preferences.

6.8 Inspecting the Mechanism

Production function in the first period

$$
\hat{y}_t = \hat{A}_t + (1-\alpha)\hat{N}_t
$$

GDP response crucially depends on the labor response. A large y response, requires very elastic labor supply (RBC requires 2-4, microevidence of intensive elasticity is 0.5).

We are richer: optimal to raise consumption today and in the future.

There is smoothing: consumption goes up, but investment go up a lot.

Permanent shock: wealth effects and substitution effects on labor supply cancel out.

Temporary shock: wealth effects smaller, substitution effects larger.

The MPL goes up, wage rises $\rightarrow$ substitution effect on labor supply.

As the persistence rises, the wealth effect becomes bigger.

We'd like to consume and enjoy more leisure.

All other things equal, the labor response is smaller.

But the leisure/consumption choice also depends on the interest rate.

6.9 The Baseline RBC Model

A baseline RBC model = Ramsey model + technology shock

Difference between the current one with the Ramsey model: the inclusion of leisure in utility, and the shocks.

An intertemporal equation links today's consumption and tomorrow's consumption.

An intratemporal equation links consumption and labor supply choice.

Assumptions to make the model analytically solvable: no government spending and fully depreciation of capital goods.

6.10 Social Planner's Problem and Decentralized Equilibrium

The two approaches are identical, because there are no market imperfections, so the first welfare theorem holds: the competitive, decentralized equilibrium is a solution to the planner problem.

6.10.1 Social Planner's Problem

$$\begin{aligned}\mathcal{L} = &\sum_{t=0}^{\infty} \beta^t u\left(c_t, L_t\right) \\ &+ \sum_{t=0}^{\infty} \beta^t \lambda_t \left[A_t F\left(k_t, N_t\right) + (1-\delta)k_t - c_t - \gamma k_{t+1}\right] \\ &+ \sum_{t=0}^{\infty} \beta^t \omega_t \left[1 - L_t - N_t\right]\end{aligned}$$

FOCs

$$\begin{aligned}\{c_t\}: &\quad u_1\left(c_t, L_t\right) = \lambda_t \\ \{L_t\}: &\quad u_2\left(c_t, L_t\right) = \omega_t \\ \{N_t\}: &\quad \lambda_t A_t F_2\left(k_t, N_t\right) = \omega_t \\ \{k_{t+1}\}: &\quad \beta\lambda_{t+1}\left[A_{t+1} F_1\left(k_{t+1}, N_{t+1}\right) + 1 - \delta\right] = \gamma\lambda_t\end{aligned}$$

Here ω is not wage w.

6.10.2 Social Planner's Problem: Bellman Equation Approach

Write planner's problem as a Bellman equation

$$\begin{aligned}V\left(A_t, k_t\right) = &\max_{c_t, N_t, k_{t+1}} u\left(c_t, N_t\right) + \beta \mathbb{E}_t V\left(A_{t+1}, k_{t+1}\right) \\ &\text{s.t.} \quad A_t k_t^{\alpha} N_t^{1-\alpha} = c_t + k_{t+1}\end{aligned}$$

7 New Keynesian Model

7.1 The Three Blocks

Three "blocks" to the model and the three equations. (IS-LM?)

Households and IS curve.

Firms and the Phillips curve.

Monetary authority and a monetary policy rule to close the model.

$$
\begin{aligned}
\tilde{y_t} &= -\frac{1}{\sigma}\mathbb{E}_t\left\{\hat{i_t} - \hat{\pi_{t+1}} - \hat{r_t^n}\right\} + \mathbb{E}_t\left\{\tilde{y_{t+1}}\right\} \\
\hat{\pi}_t &= \kappa\tilde{y}_t + \beta\mathbb{E}_t\left\{\hat{\pi_{t+1}}\right\} \\
\hat{i_t} &= \phi_\pi\hat{\pi}_t + \phi_y\tilde{y}_t + v_t
\end{aligned}
$$

7.2 Households

$$
\begin{aligned}
&\max_{C_t, N_t, B_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) Z_t \\
&\text{s.t. } C_t + Q_t\frac{B_t}{P_t} = \frac{B_{t-1}}{P_t} + \frac{W_t}{P_t}N_t + \frac{\Pi_t}{P_t}
\end{aligned}
$$

FOCs

$$
\begin{aligned}
\lambda_t &= Z_t C_t^{-\sigma} \\
\lambda_t w_t &= Z_t N_t^{\varphi} \\
Q_t = 1/R_t &= \beta\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t}\frac{P_t}{P_{t+1}}
\end{aligned}
$$

where $w_t = W_t/P_t$.

7.2.1 Consumption Varieties

C_t is the choice of the composite consumption bundle(混合消费束)

Structure of monopolistic competition follows Dixit and Stiglitz(1977). Different varieties of goods

$$
C_t = \left[\int_0^1 C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}
$$

ε is the price elasticity of demand for variety j.

7.2.2 Maximization of Consumption

$$
\begin{aligned}
&\max_{c_t(j)} \left[\int_0^1 C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}j\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&\text{s.t.} \quad X_t = \int_0^1 P_t(j)C_t(j)\mathrm{d}j
\end{aligned}
$$

FOCs

$$\{C_t(j)\}: \quad C_t(j)^{-\frac{1}{\varepsilon}} C_t^{\frac{1}{\varepsilon}} = \xi_t P_t(j)$$

Solve.

$$\frac{C_t(i)}{C_t(j)} = \left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon}$$
$$\Rightarrow C_t(j) = C_t(i)\left(\frac{P_t(i)}{P_t(j)}\right)^{\varepsilon}.$$

Plug into

$$X_t = \int_0^1 P_t(j) C_t(j) \mathrm{d}j$$
$$= C_t(i) P_t(i)^{\varepsilon} \int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j.$$

Get

$$C_t(i) = P_t(i)^{-\varepsilon} \frac{X_t}{\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j}.$$

Plug the definition of C_t

$$\begin{aligned}
C_t &= \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\int_0^1 \left(P_t(i)^{-\varepsilon} \frac{X_t}{\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j}\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \frac{X_t}{\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j} \left(\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \frac{X_t}{\left(\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j\right)^{\frac{1}{1-\varepsilon}}}.
\end{aligned}$$

of which the last equation comes from that i and j are inter-changeable in the last step. From

$$C_t = \frac{X_t}{\left(\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j\right)^{\frac{1}{1-\varepsilon}}}.$$

Combining with definition

$$X_t = P_t C_t,$$

we know that

$$P_t = \left(\int_0^1 P_t(j)^{1-\varepsilon} \mathrm{d}j\right)^{\frac{1}{1-\varepsilon}}.$$

Back again into our previous equation

$$\begin{aligned}C_t(i) &= P_t(i)^{-\varepsilon}\frac{X_t}{\displaystyle\int_0^1 P_t(j)^{1-\varepsilon}\mathrm{d}j} = P_t(i)^{-\varepsilon}\frac{X_t}{P_t^{1-\varepsilon}}\\ &= \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\frac{X_t}{P_t} = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t.\end{aligned}$$

□

7.3 Producers

Demand for variety i

$$\begin{aligned}C_t(i) &= P_t(i)^{-\varepsilon}\frac{X_t}{\displaystyle\int_0^1 P_t(j)^{1-\varepsilon}\mathrm{d}j} = P_t(i)^{-\varepsilon}\frac{X_t}{P_t^{1-\varepsilon}}\\ &= \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\frac{X_t}{P_t} = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t.\end{aligned}$$

Produce variety CRS with labor

$$Y_t(i) = A_t N_t(i).$$

Output must equal consumption for each variety

$$Y_t(i) = C_t(i).$$

7.3.1 Factor Demand

Firms demands labor to minimize cost subject to production constraint

$$\min_{N_t(i)} w_t N_t(i) + \phi_t\left(Y_t(i) - A_t N_t(i)\right).$$

ϕ_t is real marginal cost, w is the real wage.

$$\phi_t = mc_t = w_t/A_t.$$

Real marginal cost is the same for all firms.

7.3.2 Calvo Assumption

Calvo (1983) pricing assumption: Each firm resets price each period with i.i.d. probability $1-\theta$.
By L.L.N.(大数定律), fraction that reset is $1-\theta$ and fraction constant is θ.
Average price duration follows geometric distribution with mean duration $\dfrac{1}{1-\theta}$.
The world not Calvo, but it could be a decent approximation.

Literature on "menu cost" models where there is an inaction region due to fixed cost of changing price.

Initial literature: Much more flexible than Calvo.

Recent literature: To match micro-pricing facts, need large and infrequent firm-level MC shocks, which looks close to Calvo.

7.3.3 The Price Decision

Firms that adjust prices choose $P_t(i), Y_t(i), N_t(i)$ to maximize expected discounted profits and demand.

Firms takes into account the possibility they will have to keep this price in the future. (Different with simple RBC)

For those who re-optimize [there are other terms in firms' objective function omitted here for they are not related to $P_t(i)$]

$$\max_{P_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \beta^s \frac{\lambda_{t+s}}{\lambda_t} \left[\frac{P_t(i)}{P_{t+s}} Y_{t+s}(i) - \phi_{t+s} Y_{t+s}(i) \right]$$

$$\text{s.t. } C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t \quad \text{and} \quad C_t(i) = Y_t(i) \quad \text{and} \quad C_t = Y_t$$

Solve it, we get:

If $\theta = 0$, no stickiness and this collapse to flex price model.

$$P_t^* = (1+\mu)\phi_t P_t, \quad \mu = \frac{1}{\varepsilon - 1}.$$

If $\theta > 0$,then the optimal reset price is a markup over a weighted average of expected future marginal costs

$$P_t^* = (1+\mu) \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\varepsilon-1} P_{t+s} \phi_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\varepsilon-1}}.$$

Re-define $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ is the stochastic discount factor.

7.3.4 Price Dynamics with Calvo

Assume symmetric model, so fraction $1-\theta$ of firms adjust to P_t^* and fraction θ keep $P_{t-1}(i)$.

$$
\begin{aligned}
P_t &= \left[\int_0^1 P_t(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}} \\
&= \left[\theta\int_0^1 P_{t-1}(i)^{1-\varepsilon}\mathrm{d}i + (1-\theta)\int_0^1 \left(P_t^*\right)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}} \\
&= \left[\theta\left[\int_0^1 P_{t-1}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1-\varepsilon}{1-\varepsilon}} + (1-\theta)\int_0^1 \left(P_t^*\right)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}} \\
&= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)\left(P_t^*\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.
\end{aligned}
$$

Price index P_t is geometric average of P_t-1 and P_t^*.

Recursive formulation is part of why Calvo is so tractable.

7.3.5 Inflation Dynamics with Calvo

Divide by P_{t-1} to get inflation between $t-1$ and t, Π_t

$$
\Pi_t = \frac{P_t}{P_{t-1}} = \left[\theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.
$$

Calvo pricing implies a partial adjustment mechanism

If $P_t^* = P_{t-1}$, $\Pi_t = 1$.

If $P_t^* > P_{t-1}$, $\Pi_t > 1$ and $P_t \neq P_{t-1}$.

7.4 Completing The Model

Aggregate output is

$$
\begin{aligned}
N_t &= \int_0^1 N_t(i)\mathrm{d}i \\
N_t &= \int_0^1 \frac{Y_t(i)}{A_t}\mathrm{d}i = \frac{Y_t}{A_t}\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\mathrm{d}i. \\
Y_t &= A_t N_t \frac{1}{\displaystyle\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\mathrm{d}i}.
\end{aligned}
$$

Term in fraction is loss in output due to misallocation caused by price dispersion.

Creates welfare costs of inflation, but it is second order and drops out of log-linearization.

7.4.1 A Slightly Detour: The Labor Wedge

Because of CRS, nominal marginal cost is

$$MC_t = \frac{W_t}{Y_t(i)/N_t(i)} = \frac{W_t}{A_t}$$
$$mc_t = \frac{MC_t}{P_t}.$$

The labor wedge

Note that for firm with markup $\mu_t(i)$

$$1 + \mu_t(i) = \frac{P_t(i)/P_t}{mc_t}.$$

In flexible price equilibrium, $P_t(i)/P(t) = 1$, so

$$1 + \mu_t(i) = 1/mc_t = \frac{Y_t/N_t}{W_t/P_t} = \frac{MPL_t}{MRS_t}.$$

7.5 New Keynesian Model Equilibrium

Definition A symmetric equilibrium is an allocation $\{C_{t+s}, N_{t+s}, Y_{t+s}\}_s^\infty = 0$ and set of prices $\{P^*_{t+s}, P_{t+s}, W_{t+s}, Q_{t+s}\}_s^\infty = 0$ along with exogenous processes $\{A_{t+s}, Z_{t+s}, v_{t+s}\}_s^\infty = 0$ such that:

1. Households optimize: Euler, labor-leisure, (money demand in background as the central bank chooses Q_{t+s}, putting B_{t+s} in background as well).

2. Firm optimize: Price index follows dynamic Calvo formulation; Intermediate reset prices are chosen optimally given nominal marginal cost:$MC_t = \dfrac{W_t}{Y_t(i)/N_t(i)} = W_t A_t$.

3. Central bank follows interest rate rule with shock v_t.

4. Labor and goods (and bond) markets clear.

7.6 Log Linearization Strategy

Phillips curve should be function of output gap, so want to write whole model as functions of output gap.

Strategy:

1. Log-linearize Model around zero-inflation steady state.(IS-PC-MP)
2. Log-linearize Flexible Price Equilibrium.
3. Difference to get equilibrium in terms of output gap.

7.6.1 Supply Block

$$
\begin{aligned}
\frac{W_t}{P_t} &= \frac{N_t^{\varphi}}{C_t^{-\sigma}} \\
Y_t &= A_t N_t \frac{1}{\displaystyle\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \mathrm{d}i}. \\
\Pi_t &= \frac{P_t}{P_{t-1}} = \left[\theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}. \\
P_t^* &= (1+\mu)\frac{\mathbb{E}_t \displaystyle\sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\varepsilon-1} P_{t+s}\phi_{t+s}}{\mathbb{E}_t \displaystyle\sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\varepsilon-1}}.
\end{aligned}
$$

Labor-leisure

$$\hat{w}_t - \hat{p}_t = \varphi \hat{n}_t + \sigma \hat{c}_t.$$

Production function: [one term missing here,see Appendix]

$$\hat{y}_t = \hat{a}_t + \hat{n}_t.$$

7.6.2 Inflation and Reset Prices

Details in appendix.

Key trick: Zero inflation steady state.

The price index can be log-linearized to get

$$
\begin{aligned}
\hat{p_t} &= \theta \hat{p_{t-1}} + (1-\theta)\hat{p_t^*}. \\
\hat{\pi}_t &= (1-\theta)\left(\hat{p_t^*} - \hat{p_{t-1}}\right).
\end{aligned}
$$

7.6.3 Phillips Curve

We will get an expectation-augmented Phillips curve [see appendix for details]

$$\hat{\pi}_t = \lambda \hat{\phi}_t + \beta \mathbb{E}_t \{\pi_{t+1}\}.$$

where

$$\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}.$$

Inflation is equal to expected future inflation plus the deviation of marginal cost from its steady state level.

Iterating forward

$$\hat{\pi}_t = \lambda \mathbb{E}_t \left\{\sum_{s=0}^{\infty} \beta^s \hat{\phi}_{t+s}\right\} = \lambda \mathbb{E}_t \left\{\sum_{s=0}^{\infty} \beta^s \hat{m}c_{t+s}\right\}.$$

Inflation is the present discounted value of future marginal cost/markup deviations from steady state.

7.6.4 Real Marginal Costs

Real marginal cost

$$\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{a}_t.$$

Combine labor-leisure, production function, and $\hat{c}_t = \hat{y}_t$

$$\hat{w}_t - \hat{p}_t = (\sigma + \varphi)\hat{y}_t - \varphi\hat{a}.$$

Consequently

$$\hat{mc}_t = (\sigma + \varphi)\hat{y}_t - (1 + \varphi)\hat{a}_t.$$

7.6.5 Flexible Price Equilibrium

$$\begin{aligned}
Y_t^n &= A_t N_t^n \\
\frac{W_t^n}{P_t^n} &= \frac{A_t}{1+\mu} \\
\frac{W_t^n}{P_t^n} &= \frac{(N_t^n)^{\varphi}}{(C_t^n)^{-\sigma}} \\
Y_t^n &= C_t^n
\end{aligned}$$

Combine to get

$$\begin{aligned}
A_t^{1+\varphi} &= (1+\mu)\left(Y_t^n\right)^{\sigma+\varphi}. \\
(\sigma + \varphi)\hat{y}_t^n &= (1 + \varphi)\hat{a}_t.
\end{aligned}$$

Combine

$$\begin{aligned}
\hat{mc}_t &= (\sigma + \varphi)\hat{y}_t - (1 + \varphi)\hat{a}_t \\
(\sigma + \varphi)\hat{y}_t^n &= (1 + \varphi)\hat{a}_t
\end{aligned}$$

to write real marginal costs in terms of output gap $\tilde{y}_t$

$$\hat{mc}_t = (\sigma + \varphi)\left(\hat{y}_t - \hat{y}_t^n\right) = (\sigma + \varphi)\tilde{y}_t.$$

Real marginal costs go up (and markups go down) when the output gap is high.
To produce more than under flex prices, markup must be lower.
Marginal costs high because need to hire more workers, bidding up real wage.
Stronger when Intertemporal Elasticity of Substitution and labor supply elasticity are low.

7.7 The New Keynesian Phillips Curve

Plug back into the Phillips curve $\hat{\pi}_t = \lambda \hat{mc}_t + \beta \mathbb{E}_t \{\pi_{t+1}\}$

$$\hat{\pi}_t = \kappa \tilde{y}_t + \beta \mathbb{E}_t \{\hat{\pi_{t+1}}\}.$$

where $\kappa = \lambda(\sigma + \varphi)$.

This is the **New Keynesian Phillips Curve**: an expectations augmented Phillips curve written in terms of the output gap.

Solving forward

$$\hat{\pi}_t = \kappa \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \tilde{y_{t+s}}.$$

Inflation is an increasing function of future output gaps.

7.8 The New Keynesian IS Curve

7.8.1 Demand

$$Q_t = 1/R_t = \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}}$$

$$Y_t = C_t$$

Log-linearize Euler around zero-inflation

$$\hat{c_t} = -\frac{1}{\sigma}\left(\hat{i_t} - \mathbb{E}_t \{\hat{\pi_{t+1}}\} - (\hat{z_t} - \mathbb{E}_t \hat{z_{t+1}})\right) + \mathbb{E}_t \{\hat{c_{t+1}}\}.$$

Combine with market clearing and use

$$\hat{y_t} = -\frac{1}{\sigma}\left(\hat{i_t} - \mathbb{E}_t \{\hat{\pi_{t+1}}\} - (\hat{z_t} - \mathbb{E}_t \hat{z_{t+1}})\right) + \mathbb{E}_t \{\hat{y_{t+1}}\}.$$

This is the dynamic IS curve. It relates output to future expectations of output and the real interest rate.

7.8.2 The Natural Rate of Interest

Define the natural rate of interest $\hat{r_t}^n$ as the real interest rate that would prevail when output is equal to its flexible level(natural level of output)

$$\hat{y_t^n} = -\frac{1}{\sigma}\left(\hat{r}_t^n - (\hat{z_t} - \mathbb{E}_t \hat{z_{t+1}})\right) + \mathbb{E}_t \left\{\hat{y_{t+1}^n}\right\}.$$

Recall $\hat{y_t^n} = \left(\frac{1+\varphi}{\sigma+\varphi}\right) \hat{a}_t$ so

$$\hat{r_t^n} = \sigma \left(\frac{1+\varphi}{\sigma+\varphi}\right) \mathbb{E}_t \{\hat{a_{t+1}} - \hat{a}_t\} + (\hat{z_t} - \mathbb{E}_t \hat{z_{t+1}}).$$

If at follows an $AR(1)$ and grows today, it will be expected to decline between today and tomorrow due to mean reversion.

So positive tech shock causes real interest rate to fall by standard RBC logic.

7.8.3 IS Curve in Terms of the Output Gap

$$\hat{y_t} = -\frac{1}{\sigma}\left(\hat{i_t} - \mathbb{E}_t\left\{\hat{\pi_{t+1}}\right\} - \left(\hat{z_t} - \mathbb{E}_t\hat{z_{t+1}}\right)\right) + \mathbb{E}_t\left\{\hat{y_{t+1}}\right\}.$$
$$\hat{y}_t^n = -\frac{1}{\sigma}\left(\hat{r_t}^n - \left(\hat{z_t} - \mathbb{E}_t\hat{z_{t+1}}\right)\right) + \mathbb{E}_t\left\{\hat{y}_{t+1}^n\right\}.$$

So, we have

$$\tilde{y_t} = -\frac{1}{\sigma}\mathbb{E}_t\left\{\hat{i_t} - \hat{\pi_{t+1}} - \hat{r_t^n}\right\} + \mathbb{E}_t\left\{\tilde{y_{t+1}}\right\}.$$

Iterating forward, the current output gap depends negatively on the gap between the real interest rate and the natural rate of interest

$$\tilde{y_t} = -\frac{1}{\sigma}\mathbb{E}_t\left\{\sum_{s=0}^{\infty}\left(\hat{r_{t+s}} - \hat{r}_{t+s}^n\right)\right\}.$$

7.9 Summary

7.9.1 The Three Equations Model

In sum, the NK model boils down to three equations

$$\begin{aligned}
\tilde{y_t} &= -\frac{1}{\sigma}\mathbb{E}_t\left\{\hat{i_t} - \hat{\pi_{t+1}} - \hat{r_t^n}\right\} + \mathbb{E}_t\left\{\tilde{y_{t+1}}\right\} \\
\hat{\pi}_t &= \kappa\tilde{y}_t + \beta\mathbb{E}_t\left\{\hat{\pi_{t+1}}\right\} \\
\hat{i_t} &= \phi_\pi\hat{\pi}_t + \phi_y\tilde{y}_t + v_t
\end{aligned}$$

with three unknowns: $\hat{i_t}$, $\tilde{y}_t$, and $\hat{\pi}_t$ and an exogenous driving process for the natural rate

$$\hat{r}_t^n = \sigma\left(\frac{1+\varphi}{\sigma+\varphi}\right)\mathbb{E}_t\left\{\hat{a_{t+1}} - \hat{a}_t\right\} + \left(\hat{z_t} - \mathbb{E}_t\hat{z_{t+1}}\right).$$

7.9.2 Limitation of "Simple" NK Models

"Simple" New Keynesian models like the one we have studied have limitations.

For starters, no capital! No inflation persistence. No wage stickiness (which makes marginal costs sticky and prices stickier). Little amplification of Calvo friction (price level has adjusted once the time most firms have reset price).

Big literature adding many features back into basic NK model.

$\beta = 0.99$

$\alpha = 0.33$ (Back to more general case). $\sigma = 1$ (Log utility)

$\phi = 5$ (Frisch elasticity of labor supply: 0.2)

$\epsilon = 9$ (Steady state markup: 12.5 percent)

$\theta = 0.75$ (Average price duration: four quarters)

$\phi_\pi = 1.5, \phi_y = 0.125.$

$\rho_v = 0.5, \rho_z = 0.5, \rho_a = 0.9$

Figure 16 Calibration Parameters

7.10 Appendix

7.10.1 Connecting Inflation with The Optimal Reset Price

We need to log-linearize the following equation connecting the current price index with the lagged price index and the optimal reset price.

$$\frac{P_t}{P_{t-1}} = \left[\theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

$$\left(\frac{P_t}{P_{t-1}}\right)^{1-\varepsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon}.$$

Take the total derivative around the steady state

$$(1-\varepsilon)\left(\frac{P_t}{P_{t-1}}\right)^{-\varepsilon}(\mathrm{d}p_t - \mathrm{d}p_{t-1}) = (1-\theta)(1-\varepsilon)\left(\frac{P_t^*}{P_{t-1}}\right)^{-\varepsilon}(\mathrm{d}p_t^* - \mathrm{d}p_{t-1}).$$

since we are at the steady state, the term in blue equals one.

$$p\hat{p}_t - p\widehat{p_{t-1}} = (1-\theta)\left(p\widehat{p_t^*} - p\widehat{p_{t-1}}\right).$$

in which we use the definition of $\hat{x_t} = \frac{x_t - x}{x} = \frac{\mathrm{d}x_t}{x}$. x is the steady state value for a certain variable. Then we get

$$\hat{\pi_t} = (1-\theta)\left(\widehat{p_t^*} - \widehat{p_{t-1}}\right).$$

7.10.2 Price Dispersion in Production

We need to log-linearize the following term

$$Y_t = A_t N_t \frac{1}{\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \mathrm{d}i}.$$

we write in the following form

$$\hat{y_t} = \hat{a_t} + \hat{n_t} - \hat{d_t}.$$

of which d_t corresponds to the price dispersion term.

We expand the exponential function around the steady state where$P_t(i) = p_t$.

$$\begin{aligned}
\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} &= \exp\left[(1-\varepsilon)\left(\hat{p_t((i)} - \hat{p}_t\right)\right] \\
&= 1 + (1-\varepsilon)\left(\hat{p_t(i)} - \hat{p}_t\right) + \frac{(1-\varepsilon)^2}{2}\left(\hat{p}_t(i) - \hat{p}_t\right)^2.
\end{aligned}$$

Since we have $\displaystyle\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \mathrm{d}i = 1$. A second order linear approximation will lead to

$$\begin{aligned}
\mathbb{E}_i\left\{\hat{p_t(i)} - \hat{p}_t\right\} &= \frac{\varepsilon-1}{2}\mathbb{E}_t\left\{\left(\hat{p_t(i)} - \hat{p}_t\right)^2\right\}. \\
\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} &= 1 - \varepsilon\left(\hat{p_t(i)} - \hat{p}_t\right) + \frac{\varepsilon^2}{2}\left(\hat{p_t(i)} - \hat{p}_t\right)^2.
\end{aligned}$$

Combine the above two equations

$$\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \mathrm{d}i = 1 + \frac{\varepsilon}{2}\mathbb{E}_t\left\{\left(\hat{p_t(i)} - \hat{p}_t\right)^2\right\} \approx 1 + \frac{\varepsilon}{2}\operatorname{var}_i\left\{\hat{p_t(i)}\right\}.$$

The second line comes from

$$\int_0^1 \left(\hat{p_t(i)} - \hat{p}_t\right)^2 \mathrm{d}i \approx \int_0^1 \left(\hat{p}_t(i) - \mathbb{E}_i\left\{\hat{p}_t\right\}\right)^2 \mathrm{d}i \triangleq \operatorname{var}_i\left\{\hat{p}_t(\hat{i})\right\}.$$

So

$$d_t = \ln\left[\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \mathrm{d}i\right] \approx \frac{\varepsilon}{2}\operatorname{var}_i\left\{\hat{p_t(i)}\right\}.$$

7.10.3 Linearization of The Optimal Price Setting

We know

$$P_t^* = (1+\mu)\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\theta^s\Lambda_{t,t+s}Y_{t+s}P_{t+s}^{\varepsilon-1}P_{t+s}\phi_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}\theta^s\Lambda_{t,t+s}Y_{t+s}P_{t+s}^{\varepsilon-1}}.$$

Rewrite the denominator and numerator

$$\begin{aligned}
X_{1,t} &= Z_t C_t^{-\sigma} mc_t P_t^{\varepsilon} Y_t + \beta\theta\mathbb{E}_t X_{1,t+1}. \\
X_{2,t} &= Z_t C_t^{-\sigma} P_t^{\varepsilon-1} Y_t + \beta\theta\mathbb{E}_t X_{2,t+1}.
\end{aligned}$$

Define

$$x_{1,t} \triangleq \frac{X_{1,t}}{P_t^{\varepsilon}} \qquad x_{2,t} \triangleq \frac{X_{2,t}}{P_t^{\varepsilon-1}}.$$

We have

$$
\begin{aligned}
x_{1,t} &= Z_t C_t^{-\sigma} mc_t Y_t + \beta\theta \mathbb{E}_t \frac{X_{1,t+1}}{P_{t+1}^{\varepsilon}} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon}. \\
x_{2,t} &= Z_t C_t^{-\sigma} Y_t + \beta\theta \mathbb{E}_t \frac{X_{2,t+1}}{P_{t+1}^{\varepsilon-1}} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon-1}.
\end{aligned}
$$

In short

$$
\begin{aligned}
x_{1,t} &= Z_t C_t^{-\sigma} mc_t Y_t + \beta\theta \mathbb{E}_t x_{1,t+1} \left(1+\pi_{t+1}\right)^{\varepsilon}. \\
x_{2,t} &= Z_t C_t^{-\sigma} Y_t + \beta\theta \mathbb{E}_t x_{2,t+1} \left(1+\pi_{t+1}\right)^{\varepsilon-1}.
\end{aligned}
$$

We use $Y_t = C_t$ and take total derivative

$$
\begin{aligned}
x_1 \hat{x_{1,t}} = &(1-\sigma) Z Y^{-\sigma} mc Y \hat{Y}_t + Z Y^{1-\sigma} mc \hat{mc}_t + Y^{1-\sigma} mc Z \hat{Z}_t \\
&+ \varepsilon\beta\theta(1+\pi)^{\varepsilon-1} x_1 (1+\pi) \mathbb{E}_t \hat{\pi_{t+1}} + \beta\theta(1+\pi)^{\varepsilon} x_1 \mathbb{E}_t \hat{x_{1,t+1}}.
\end{aligned}
$$

of which the percentage deviation of inflation is different from variables like y.

$$
\begin{aligned}
\ln\left(1+\pi_{t+1}\right) - \ln(1+\pi) &= \ln\left(1 + \frac{1+\pi_{t+1} - (1+\pi)}{1+\pi}\right) \\
&\approx \frac{\pi_{t+1} - \pi}{1+\pi} = \hat{\pi_{t+1}}.
\end{aligned}
$$

We simplify equations at zero inflation steady state$Z = 1$.

$$
\begin{aligned}
x_1 \hat{x_{1,t}} = &(1-\sigma) Y^{-\sigma} mc Y \hat{Y}_t + Y^{1-\sigma} mc \hat{mc}_t + Y^{1-\sigma} mc \hat{Z}_t \\
&+ \varepsilon\beta\theta x_1 \mathbb{E}_t \pi_{\hat{t}+1} + \beta\theta x_1 \mathbb{E}_t x_{1,\hat{t}+1}.
\end{aligned}
$$

and evaluate the third equation at the steady state we get

$$
x_1 = \frac{Y^{1-\sigma} mc}{1-\beta\theta}.
$$

plug into the previous equation

$$
\begin{aligned}
\hat{x_{1,t}} = &(1-\sigma)(1-\beta\theta)\hat{Y}_t + (1-\beta\theta)\hat{mc}_t + (1-\beta\theta)\hat{Z}_t \\
&+ \varepsilon\beta\theta \mathbb{E}_t \hat{\pi_{t+1}} + \beta\theta \mathbb{E}_t \hat{x_{1,t+1}}.
\end{aligned}
$$

Following similar steps, we have

$$
\begin{aligned}
\hat{x_{2,t}} = &(1-\sigma)(1-\beta\theta)\hat{Y}_t + (1-\beta\theta)\hat{Z}_t \\
&+ (\varepsilon-1)\beta\theta \mathbb{E}_t \hat{\pi_{t+1}} + \beta\theta \mathbb{E}_t \hat{x_{2,t+1}}.
\end{aligned}
$$

Take the difference of the two equation

$$\hat{x_{1,t}} - \hat{x_{2,t}} = (1-\beta\theta)\hat{mc_t} + \beta\theta\mathbb{E}_t\hat{\pi_{t+1}} + \beta\theta\mathbb{E}_t\left[x_{1,\hat{t+1}} - x_{2,\hat{t+1}}\right].$$

We have the following from definition of $x_{1,t}, x_{2,t}$

$$\frac{X_{1,t}}{X_{2,t}} = P_t\frac{x_{1,t}}{x_{2,t}}.$$

Thus, we have

$$\begin{aligned}
P_t^* &= (1+\mu)P_t\frac{x_{1,t}}{x_{2,t}} \\
P_t^*/P_{t-1} &= (1+\mu)P_t/P_{t-1}\frac{x_{1,t}}{x_{2,t}} \\
1+\pi_t^* &= (1+\mu)\left(1+\pi_t\right)\frac{x_{1,t}}{x_{2,t}} \\
\hat{\pi_t^*} &= \hat{\pi}_t + \hat{x_{1,t}} - \hat{x_{2,t}}
\end{aligned}$$

We got the following equation in the main slides

$$\hat{\pi}_t = (1-\theta)\left(\hat{p_t^*} - \hat{p_{t-1}}\right) = (1-\theta)\hat{\pi_t^*}.$$

We now have

$$\begin{aligned}
\hat{\pi_t^*} &= \hat{\pi}_t + \hat{x_{1,t}} - \hat{x_{2,t}} \\
\hat{\pi}_t &= (1-\theta)\hat{\pi_t^*}
\end{aligned}$$

we get

$$\frac{\theta}{1-\theta}\hat{\pi}_t = \hat{x_{1,t}} - \hat{x_{2,t}}.$$

Plug back into previous equation, we get

$$\hat{\pi}_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{mc_t} + \beta\mathbb{E}_t\hat{\pi_{t+1}}.$$